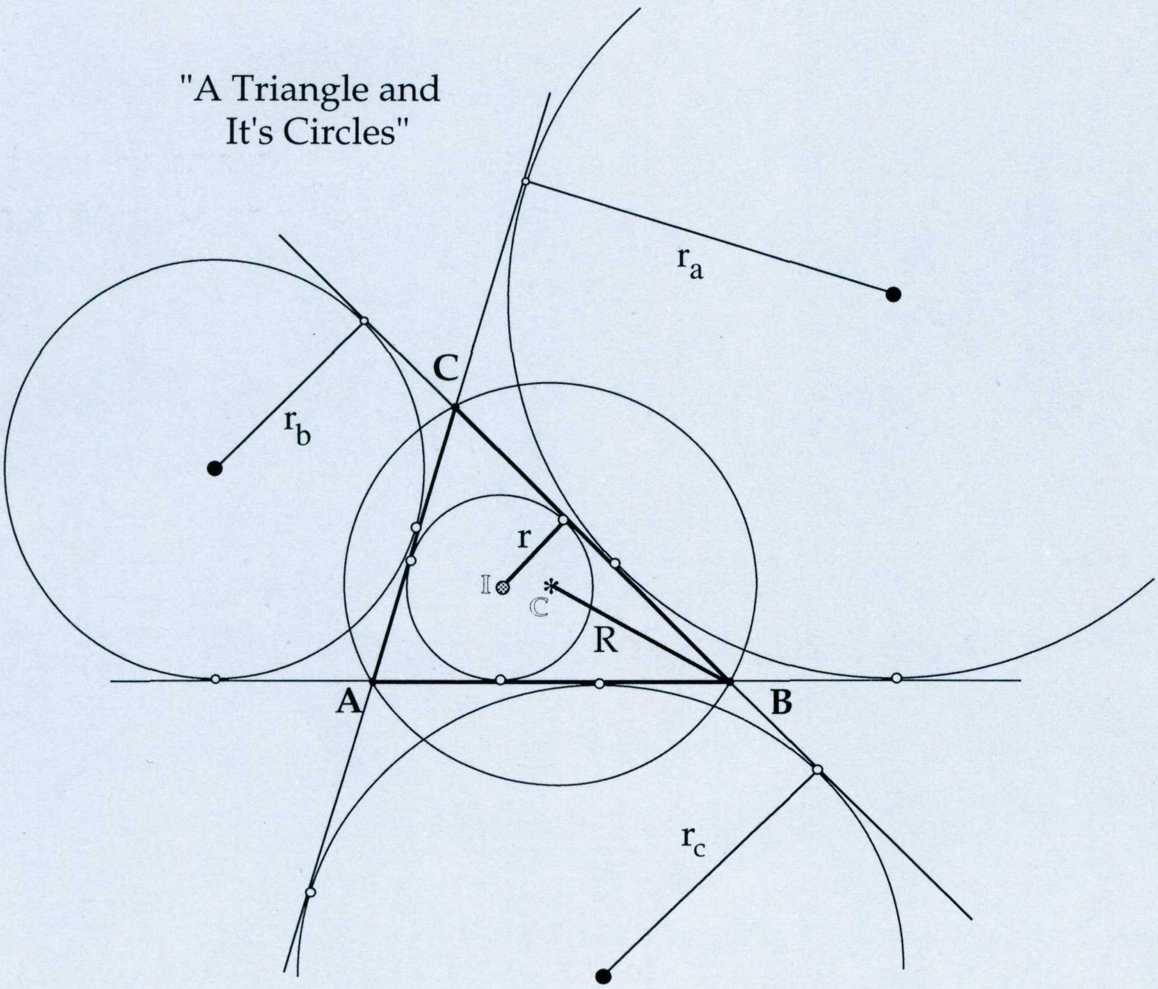
Volume IV, Number 1, Spring 1996

IMSA Math Journal

A Resource Notebook for High School Mathematics

"A Triangle and It's Circles"



An Official Publication of the Illinois Mathematics and Science Academy

IMSA

EDITORIAL POLICY

The purpose of the IMSA Math Journal is to provide a forum for high school mathematics in Illinois and nationwide. It is subtitled: "A Resource Notebook For High School Mathematics" to emphasize its intended use as a resource for teachers and students in mathematics classes.

The IMSAMJ includes problems for high school mathematics students and teachers at a variety of levels. The intention is to open the door of new ideas for exploration and discovery by both students and their teachers.

Contributions are welcome from students and teachers across the state of Illinois. Manuscripts must be typed, double spaced and written in a clear and concise style. Student materials should be proofread and corrected by a mathematics teacher at the student's home school. Manuscripts submitted to the IMSAMJ should **not** be in consideration for publication elsewhere. Since manuscripts will not be returned, please keep a copy for your own records. Once a manuscript has been accepted for publication, it becomes the property of the IMSA and may be edited or excerpted to suit journal needs.

Submission of original problems should be accompanied by clear and concise solutions with proper justifications and diagrams as needed. Solutions to problems suggested in this journal also are encouraged. Be sure to quote the problem you are solving and provide a complete explanation of the steps in your solution.

Send manuscripts, problems, solutions to problems and any other correspondence directly to any of the editors by e-mail or at:

Illinois Mathematics and Science Academy
Mathematics Department
1500 West Sullivan Road
Aurora, Illinois 60506-1000
(708) 907 - 5000 [PHONE]
(708) 907 - 5976 [FAX]

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System for Partnership Initiatives
Illinois Mathematics and Science Academy
1500 W. Sullivan Road
Aurora, Illinois 60506-1000
(708)-907-5069 or e-mail Michele Micetich at
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DUPLICATION POLICY:

IMSA is an educational laboratory for designing and testing innovative programs to transform mathematics and science teaching and learning through partnerships with teachers, schools and students in Illinois. Included in the laboratory is a residential educational program for 650 Illinois students (grades 10-12) talented in mathematics and science.

Printing of the 1996 IMSA Math Journal was made possible entirely by AT&T through its 1995 People First Gold Star Program. Through the People First Gold Star Program, grants were awarded by AT&T to Chicago area non-profit organizations and schools in three categories: traditional in-school programs, school leadership efforts and other educational programs outside the schools. A committee of AT&T employee volunteers reviewed and selected the award recipients based on the program objectives, measurement of results, impact potential, degree of education need and AT&T employee involvement and leadership.

IMSA AT&T

Volume IV, Number 1, Spring 1996

IMSA MATH JOURNAL

A Resource Notebook for High School Mathematics

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LETTER FROM THE EDITORS:

We are proud to present the fourth issue of the IMSA Math Journal. In this edition we have included work by IMSA faculty, support staff, students, and alumni as well as articles by a faculty member of a sister school, The North Carolina School of Science and Mathematics and a colleague at Eastern Illinois University.

The purpose of the IMSAMJ is to communicate through mathematics with both students and teachers. Some of our goals include:

- *Presenting teaching insights, lessons, problems.*
- *Sharing mathematical ideas, mathematical teaching ideas, observations, approaches, connections, extensions, generalizations of interest to students and teachers of mathematics.*
- *Featuring mathematics problems for use inside and outside of the classroom, i.e. math contests, math competitions, etc.*
- *Discussing and sharing the role of technology including calculators and computers in the instruction and learning of mathematics.*
- *Sharing our experiences as educators as we strive to help students construct mathematical meaning through an integrative inter- and intra-disciplinary set of learning experiences.*

We are also happy to announce the creation of an IMSA home page on the world wide web from which information about IMSA, its staff and its programs may be obtained. We are working toward providing the Math Journal *on line* and later plan to add still more activities and materials for math teachers, students and parents. In the future math team coaches and members should be able to find information about competitions and materials suitable for use in preparing for them. We are very excited about the potential for this new resource. You can find us on the web at: http://www.imsa.edu/. We sincerely appreciate the support provided to us by AT&T in funding this internet initiative as well as the publication of the Journal.

We have been encouraged by the support, kind words, and constructive suggestions we have received from readers of our previous issues of the IMSAMJ and look forward to hearing your views on this issue. Please take time to complete and return the Feedback form.

Susan Eddins, Charles Hamberg
Christina Loos, George Milauskas
Illinois Mathematics and Science Academy

IMSA

PROBLEM SETS à la IMSA

by: the Mathematics Faculty
Illinois Mathematics and Science Academy

The mathematics faculty of the Illinois Mathematics and Science Academy has developed a three semester sequence of courses, Mathematical Investigations I, II, and III, whose intent is to take students with a background in elementary algebra and geometry and help them prepare for the study of calculus. An integral part of these courses are sets of 30 - 40 problems given to students approximately once every six to seven school days.

Each *Problem Set* attempts to serve several purposes. It provides a review of material we *expect* that students have seen before but might need to bring to *present memory*, it contains problems which reinforce some of the current material, it previews constructs which will be useful in future work in mathematics, and it strives to present problems in a variety of contexts many of which link several ideas together in perhaps innovative ways - though this may be a subjective judgment.

Many of the articles which appear in the IMSA Math Journal have been taken from ideas developed for the Problem Sets. In this issue we have decided to include one Problem Set taken from the fall, 1995, Mathematical Investigations I course in its entirety. We hope you find it interesting. We would appreciate your feedback.

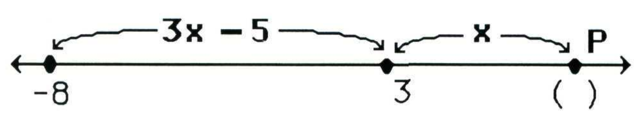
"MATHEMATICAL INVESTIGATIONS"-1

Problem Set - 4

1. The January price of a complete set of Bubs Baseball Cards went up by 20% in February. In March, the set went on sale at 20% off. By what percent did the price of a set of cards change from the January price ? (Specify increase/decrease)

2. Solve for **n**: $\dfrac{\left(a^n a^3\right)^2}{\left(a^6\right)^3} = a^{13}$

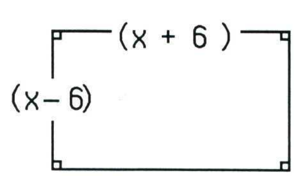
3. Find the coordinate of the point P, to the right of 3, on the number line shown.



4. Solve over the complex numbers: $2x + \frac{3}{2} = x^2$.

5. Find all integer values of x for which $\dfrac{6}{x+2}$ is an integer.

6. Find all values of **x** for which the area of the rectangle is less than 64. Write your answer in the form: $a < x < b$



7. Graph the set of ordered pairs, $(\mathbf{x},\mathbf{y})$, for which $\mathbf{x} = 6 + \mathbf{t}$ and $\mathbf{y} = 8 - 2\mathbf{t}$ for $-4 \le \mathbf{t} \le 4$ [Label the endpoints.]

8. Solve the system: $\begin{cases} 180x - 120y = 420 \\ \dfrac{x}{7} + \dfrac{y}{14} = \dfrac{1}{2} \end{cases}$ for (x,y)

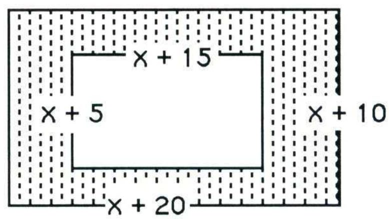
9. Recall: $n! = n(n-1)(n-2) \bullet \ldots \bullet 3 \bullet 2 \bullet 1$

Find k: a) $\dfrac{122!}{119!} = k$

b) $93! + 92! + 91! = k^2 \bullet 91!$

10. Solve for x: $x^{\frac{2}{3}} - 5 = 59$

11. Determine x, if the shaded area between the two rectangles is 875 square units.



12. Let $\mathbf{a} * \mathbf{b} = ax + by$ for all real numbers **a** and **b.**
Find (x,y), if $2 * 7 = 36$ and $(-2) * 4 = 41$.

13. P varies directly as the <u>square</u> of Q and Q = 10 when P = 6.
Find the value of P when Q = 8.

14. Solve the system: $\begin{Bmatrix} x^2 - y^2 = 63 \\ x - y = 7 \end{Bmatrix}$

15. $\square$ QUIK is transformed to $\square$ Q′U′I′K′ by 3 transformations.
$\square$ QUIK has vertices Q(8,2), U(5,4), I (0,0), K(6,–1)
$\square$ Q′U′I′K′ has vertices Q′(11,–4), U′ (8,–6), I′ (3,–2), K′ (9,–1)

a) Describe geometrically the transformations which have taken place.
b) Write the matrix equation(s) to describe the transformation action.

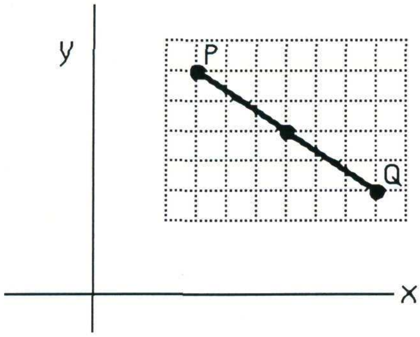
16. $C(n,r) = \dfrac{n!}{r!(n-r)!}$ Find C(92,88)

17. x is strictly within 5 units of –3.

a) Graph this set of values on a number line.

b) Write this set in set builder notation: Use the form: $\{x \mid |x-h| < r\}$

18.



Find the slope of $\overline{PQ}$.
[Careful! Scale is not given from origin.]

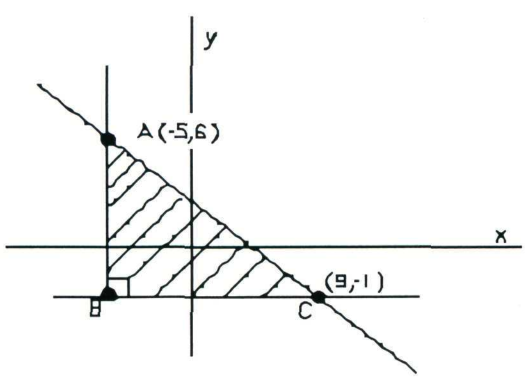
19. For demographic purposes, the U.S. Census Bureau classifies all areas of the country as being either rural, urban, or suburban. In a recent census, it was found that 27% of all adult U.S. citizens had, at some time in their lives, lived in a rural area; 74% had lived in an urban area; and 47% had lived in a suburban area. 7% of the adults had, at some time in their lives, lived in both rural and suburban areas; 27% had lived in both urban and suburban areas; and 16% had lived in both rural and urban areas.

a) What percentage of adult Americans have spent their entire lives in one single type of area (rural, urban or suburban)?

b) What percentage of adult Americans have lived in all three types of areas?

20. Find a <u>cubic</u> equation whose solutions are: $x = 0$, $x = -4$ and $x = 6$

21.



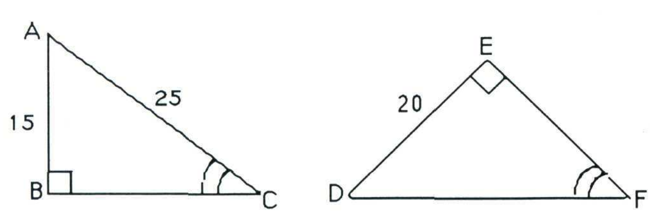
Find the equation of:

a) $\overline{AB}$

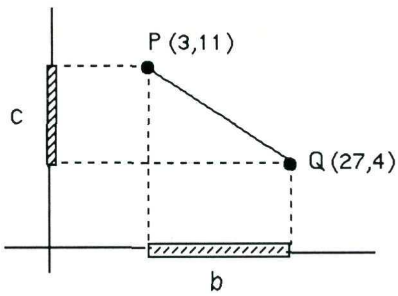
b) $\overline{BC}$

c) $\overline{AC}$

22.



Find the perimeter of ΔDEF.

23. 

a) Find the slope of $\overline{PQ}$

b) Find the shadow of $\overline{PQ}$ on the x axis. Write as a set, $\{x \mid ? \le x \le ?\}$

c) Find the shadow of $\overline{PQ}$ on the y axis. Write as a set, $\{y \mid ? \le y \le ?\}$

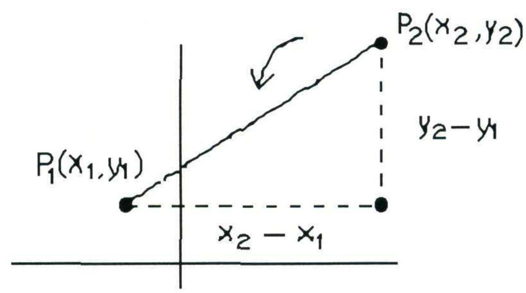
24. Define: a ® b = arithmetic mean between a and b.
a ¥ b = geometric mean between a and b.

a) Find 13 ® (9 ¥ 4) b) Find (19 ® 13) ¥ 4

c) If $a^2 + b^2 = 146$ and a ® b = 7, find a ¥ b.

25. The distance between two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

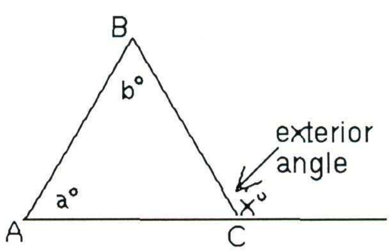


Find the perimeter of ΔFGH to the nearest hundredth if
F = (–12,–3) G = (4,7) H = (9,17)

26. The vertex matrix of ΔABC is $\begin{bmatrix} 4 & a & 10 \\ -3 & a+5 & 2 \end{bmatrix}$. Determine **a** if the area of ΔABC is 100.

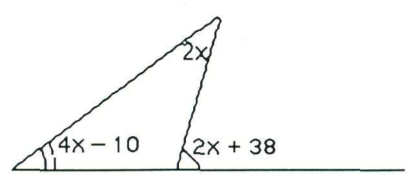
27. Simplify: $\dfrac{8}{\sqrt{2}} - 3\sqrt{8} + \dfrac{2}{3}\sqrt{288}$

28. An exterior angle of a triangle is equal to the sum of the remote interior angles.



x = a + b

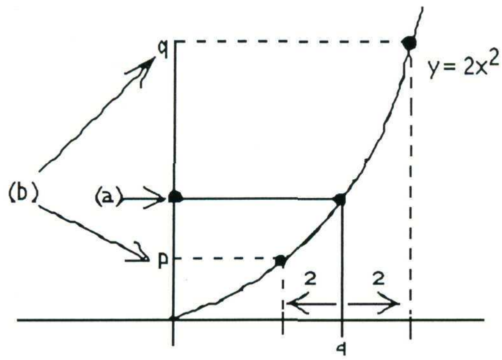
a) Solve for **x**.



b) The angles of a triangle are in the ratio of 2:3:5. Find the measure of the largest <u>exterior</u> angle.

29. $\begin{bmatrix} 6 & 2 \\ 4 & 1 \end{bmatrix} \bullet \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Find $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$

30.



a) Find y, if x = 4
b) Find p and q

31. Simplifying Radicals.

a) Multiply and simplify $(\sqrt{5} - 2)(\sqrt{5} + 2)$

b) Multiply and simplify $\dfrac{6}{(\sqrt{5} - 2)} \bullet \dfrac{(\sqrt{5} + 2)}{(\sqrt{5} + 2)}$

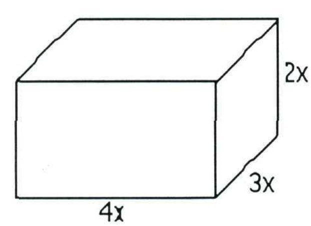
c) Rationalize denominator: $\dfrac{18}{\sqrt{7} - 4}$

32. $A = \begin{bmatrix} 5 & -3 \\ 0 & -1 \\ 6 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 4 & -3 \\ 2 & 0 & 1 \end{bmatrix}$ Find A • B and B • A

33. If $x + \dfrac{1}{x} = 5$, find $x^2 + \dfrac{1}{x^2}$

34. The edges of a rectangular box are in a ratio: length : width : height = 4 : 3 : 2

The perimeter of the smallest face is 50. Find the perimeter of the largest face.



35. Given 4 rays **(no two opposite)** with a common endpoint, 6 angles are determined.

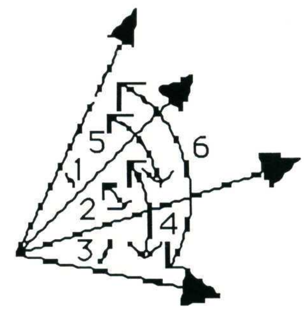
How many angles are formed by:

a) 3 rays with a common endpoint.

b) 5 rays with a common endpoint

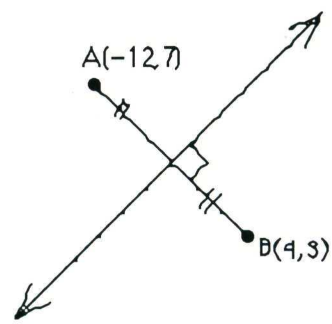
c) 6 rays with a common endpoint

d) n rays with a common endpoint



36. Solve for **x**: $\begin{vmatrix} x^2 & -6 \\ -4 & 3 \end{vmatrix} = 4 - 5x$

37. Find the equation of the line that is the perpendicular bisector of $\overline{AB}$.



38. a) Graph the line: $x - 3y = 7$

b) Find 2 ordered pairs that are integral solutions to the equation.

c) Shade the region where $x - 3y \leq 7$.

39. If Venus has a diameter of 12,100 km. while Mars has a circumference of 21,330 km., which is the **LARGER** planet? Justify.

40. Simplify:

a) $\dfrac{4! - 3!}{3!}$ b) $\dfrac{10! - 9!}{9!}$ c) $\dfrac{n! - (n-1)!}{(n-1)!}$

A commentary on selected problems:

- Problem #5 has variations which appear in other problem sets. Students are asked to find, for example, all *natural numbers* x for which the quotient is an integer or find all integer values of x for which the quotient is a *positive integer*.

- In problem #6 and again in #11 restrictions on the domain come from the fact that each side measurement as well as the area must be positive while the area is also less than 64. Students get pretty good after awhile in looking for *natural* domain restrictions in a variety of settings. Visualization enhances this process.

- Problem #9 gives students an "operational definition" for *factorial.* The goal is to increase students' reading ability in math. As a by-product, some topics are not formally "taught" in the classroom. They are introduced, defined, and practiced solely in problem sets. In addition, #9b leads to an interesting conjecture.

- We have some "favorite problem" types which are often included in problem sets. Problem #24c is a "sum-and-product" problem. Such problems give the student information on two out of three pieces of information (the sum of two numbers, their product, and the sum of their squares) and they are asked to find the missing information. An "elegant" or "clever" way to approach the problem begins by noticing that squaring the binomial relates these three expressions.

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

In #24: $$\left(\frac{a+b}{2}\right) = 7 \text{ so } (a+b) = 14 \text{ and we have } a^2 + b^2 = 146$$

$$196 = 146 + 2ab$$

$$ab = 25$$

a¥b = ±5

Later in the year students love to look for sum-and-product problems and identify all the ways we have "disguised" them in a problem set. Problem #14 looks like it might be a sum-and-product problem, but it isn't. Problem #33 is!

- Problems #23 and #30 "preview" visual constructs from calculus. Approached from the idea of "shadows" or intervals on the axes groundwork for the idea of a neighborhood can be made easily accessible. Problem #23 also relates well to later work in math or physics involving horizontal and vertical components of vectors.

- We began the year with work on geometric transformations described by matrix operations. To review those operations and see matrix operations in a new setting, they are used to present a system of equations in Problem # 29. ✍

GEOMETRIC TRANSFORMATIONS WITH MATRICES
TRANSLATIONS, ROTATIONS, REFLECTIONS, AND SCALING -- ALL AT ONCE

by: Michael Sloan
Illinois Mathematics and Science Academy

If you've been using matrices to describe geometric transformations, you've probably used matrix addition for translation (see IMSAMJ, Vol. III, #1, Spring 1995) and matrix multiplication for rotations, reflections, and scaling. Suppose you have a triangle with vertices A (0, 0), B (2, 5), C (7, –1) as shown in Figure 1.

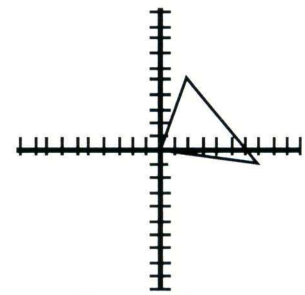


Figure 1

You could represent the triangle using a 2 x 3 matrix, with each column representing a vertex $\begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & -1 \end{bmatrix}$. Suppose you wanted to translate the figure 4 units to the left and 3 units up. You could accomplish this by *adding* the translation matrix $\begin{bmatrix} -4 & -4 & -4 \\ 3 & 3 & 3 \end{bmatrix}$ to the original vertex matrix

$$\begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & -1 \end{bmatrix} + \begin{bmatrix} -4 & -4 & -4 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 3 \\ 3 & 8 & 2 \end{bmatrix} \text{ (Figure 2)}$$

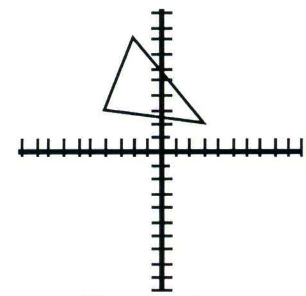


Figure 2

All the other transformations employ a 2 x 2 matrix which is used to *multiply* the vertex matrix. For example, if you want to reflect the triangle over the x-axis, you would multiply the vertex matrix by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 7 \\ 0 & -5 & 1 \end{bmatrix} \text{(Figure 3).}$$

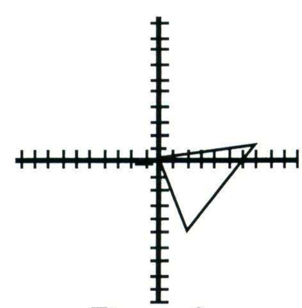


Figure 3

Similarly, to get a 90° counterclockwise rotation of the original triangle about the origin, you would multiply the vertex matrix by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 1 \\ 0 & 2 & 7 \end{bmatrix} \text{(Figure 4).}$$

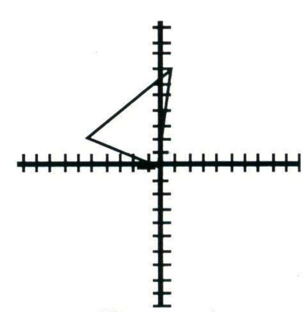


Figure 4

One last example, suppose we wanted to change the scale of our triangle. Let's double its size.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 14 \\ 0 & 10 & -2 \end{bmatrix}$$

While this approach provides a good introduction linking geometric transformations and matrices, a more "standard" and efficient approach is possible. It's not necessary, to use two different matrix operations (addition and multiplication) for transformations. One matrix can do it all!

First, instead of representing the triangle with a 2 x 3 matrix, we'll use a 3 x 3 matrix and fill in the third row with ones, so that our vertex matrix looks like: $\begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

Next, we'll use the 3 x 3 identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as the "shell" for all of our transformations.

Now, suppose we again want to translate the triangle 4 units left and 3 units up. Translations are handled by the top two positions in the *last* column -- the top position controls the movement in the x-dimension while the second position controls movement in the y-dimension. In our example, the transformation matrix looks like: $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. To transform the vertex matrix, we multiply it by the transformation matrix.

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 3 \\ 3 & 8 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Again, the top row of the resulting matrix gives the x-coordinates of the vertex points; the second row gives their y-coordinates.

All the other transformations (rotations, reflections, etc.) use the upper left 2 x 2 portion of the transformation matrix. For example, to rotate our triangle 90° counterclockwise, our transformation matrix would look like: $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. And

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 1 \\ 0 & 2 & 7 \\ 1 & 1 & 1 \end{bmatrix}$$

And yes, we can combine translations with other types of transformations. See if you can figure out what this transformation matrix will do to our triangle.

$$\begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

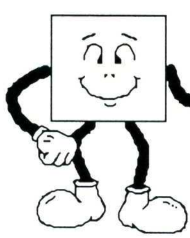
Write a matrix which would translate the triangle 2 units up, 7 units left, and reflect it over the line y = x. ✍

MATH WORDS

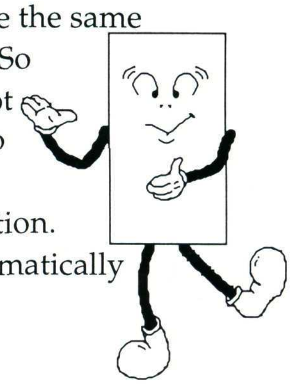
by: Dr. Micah Fogel and Christina Loos
Illinois Mathematics and Science Academy

In mathematics, we use a large and very specialized vocabulary. In college mathematics and beyond, most of the words are unique to mathematics, and this is true of many words in elementary and high school mathematics as well (quadratic, perpendicular, integer). On the other hand, most math words have at least one common meaning as well (slope, intersection, variable). Indeed, the word "set" is listed by the Guiness Book of World Records to have more definitions than any other word in the English language. This is especially interesting in that mathematics considers "set" to be an undefined term.

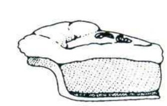
Mathematics is a science. Science requires precision. So we must be very careful about how we use our math words, to make sure that we are not distorting their meanings. If we are not perfectly clear about what we are saying mathematically, our statements will probably turn out to be incorrect, if not downright meaningless. It is not too hard to use math-only words correctly, as long as we pay attention to the definitions. But words that have common meaning that have been borrowed *by* mathematics, or that have been borrowed *from* mathematics, need extra care.

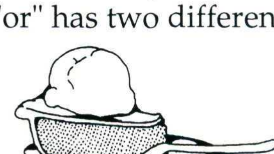


A simple example: are a square and a rectangle *similar* shapes? If you ask this question outside of a mathematical context, the answer will probably be "yes". They both have the same number of sides and right angles at all the corners. So they're close. Are they mathematically similiar? Not at all. Mathematical similarity requires one shape to be an enlargement or reduction of the other shape, with the same stretching or shrinking in every direction. The only way a rectangle and a square can be mathematically similar is if the rectangle is itself a square.

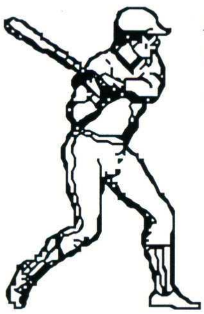


For dessert, you can have pie *or* ice cream. Can you have both? In common English, you cannot. "Or" is exclusive, forbidding both of the things it joins from being true at the same time. Mathematically, "or" is inclusive. The math statement "pie or ice cream" is the same as the English statement "pie or ice cream or both". The context in which you are speaking makes a very big difference when such a common word as "or" has two different meanings!





Usually, you can guess the basic meaning of a math word, if not its exact definition, from its meaning in English. Thus, the *slope* is a measure of how much slant a line has, a *union* is a bunch of things joined together, and a *median* is something that is in the middle. Often, however, words are borrowed without their normal meaning, or with the meaning specially altered to fit a particular use.



An example of this would be *sequence* and *series*. In English, both of these words mean essentially the same thing: a bunch of things in a particular order. Mathematics has borrowed "sequence" in this meaning, but reserves a special meaning for "series". A series in mathematics is where the things in a sequence are added together. So " 1, 2, 3, 4" is sequence, but is not a series because the terms are not summed. Similarly, if left to mathematicians, we would have to rename the baseball championship "the World Sequence" since the games are not added together. Good thing it wasn't left to mathematicians, though; "the World Sequence" just doesn't have the right ring.

One final example. Does your TV or VCR have a 37- function remote (or whatever the number is)? Mine does. In English, a "function" is a use or operation. My TV remote control has 37 buttons, each of which makes the TV do something different, so 37 different operations are performed. By the mathematical definition of a function, none of those 37 operations are functions. In mathematics, if you have a function and put the same thing in each time, the same thing has to come out. On my remote, if I press the "volume down" button, the volume doesn't always go down after it has been reduced to the minimum it can't go down any further. Mathematically, I would describe my remote control as a single function which has two inputs (the current state of the TV and the button that I press) and gives one output (the new state of the TV). For instance, each time I press the "channel up" button when the TV is on channel 2, it goes to channel 3.

People are very good at figuring out the meaning of things that are ambiguous. Usually it will be clear whether you are using a word in its mathematical context or its normal usage. Usually the audience can fill in the missing pieces when you fail to be specific with your vocabulary. When you ask someone to rotate an object, you don't always have to specify the axis of rotation and the direction and angle through which you want the object rotated. But why make things hard on your audience? When you write or speak you strive to transmit your ideas quickly, efficiently, and accurately. Mathematical ideas should not be an exception. If used correctly and carefully, mathematical language is very efficient and accurate. You just have to think!

Activities:

- Make a list of math vocabulary words you have used recently. Are they unique to mathematics, borrowed from English, or did English borrow them from mathematics? What is the difference in their math and common usages? What is their exact mathematical meaning, and what things do you need to know to use them properly?

- Try to find a mathematical dictionary that gives etymologies. Where did some of your words come from? Some words have very interesting histories (try finding out about algebra, algorithm, cipher, imaginary number, fractal, and Cartesian coordinates). Where did some mathematical conventions come from, such as using x for a variable, m for the slope, e for the base of the natural logarithm, and π for the ratio of the circumference to the diameter?
- Try to think of examples where using a word without being careful about its context can get you into trouble.
- Bring in examples of math words that are used in common situations. Do they have meanings which are similar to their mathematical meanings? Or is it like the 37-function remote? How would you describe the situation properly from a mathematical perspective?
- Watch your homework, writing, and speech that includes mathematics. Are you always as careful as you should be when using math words? ✍

THE *PALINDROME ORDER* OF A NUMBER

by: Susan K. Eddins
Illinois Mathematics and Science Academy

A palindromic number
is any number which has the same value when read from either direction.

Try this:
- Write down any 3 digit number
- Under it write down the number you get by reversing the digits in your original number.
- Add the two numbers.
- If the number is a palindrome, stop. If the number is not a palindrome, under it write down the number you get by reversing its digits.
- Add these two numbers.
- Repeat this process until you get a palindrome.
- Count the number of times that you had to add in order to reach the palindrome. That number is the "palindrome order" of the number you started with.

Some examples:

$$\begin{array}{r} 423 \\ +\ \underline{324} \\ 747 \end{array}$$

So 423 has a *palindrome order* of 1.

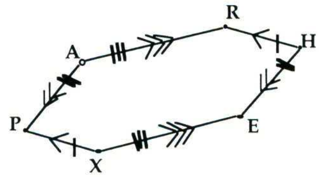
$$\begin{array}{r} 4782 \\ +\ \underline{2874} \\ 7656 \\ +\ \underline{6567} \\ 14223 \\ +\ \underline{32241} \\ 46464 \end{array}$$

So 4782 has a *palindrome order* of 3.

See how many 3-digit numbers of *palindrome order* 1 you can find.
See how many 4-digit numbers of *palindrome order* 1 you can find.
What has to be true about any number with *palindrome order* 1?
What is the largest three digit number with a *palindrome order* of 1?
See how many numbers of *palindrome order* 3 you can find.
What would it mean for a number to have a *palindrome order* of 0?
Find a number with a *palindrome order* of 10 or more. ✍

PARAHEXES

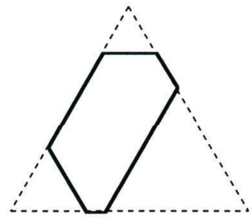
by: Barry Schnorr, Class of 1998
Illinois Mathematics And Science Academy



Parahexes are hexagons that share some properties of a parallelogram. (I thought that "parahexus" was a shorter and better name than "parallelohexagon")

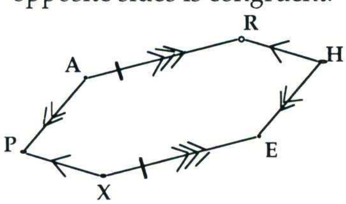
Definition: a *parahexus* is a hexagon with all three pairs of opposite sides both parallel and congruent.

The word "both" in this definition is important. In a hexagon opposite sides being congruent does not imply that they are parallels and parallel opposite sides need not be congruent. For example, take a equilateral triangle with smaller equilateral triangles lopped off the corners. While opposite sides are parallel, such a figure is not a parahexus.

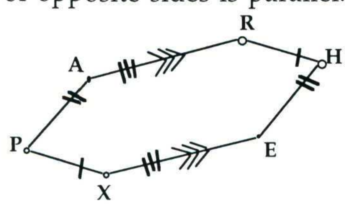


In order to prove that a hexagon is a parahexus, one does not have to prove *all* pairs of opposite sides both parallel and congruent. In fact, as an extension of the methods used for parallelograms, you can prove that the hexagon is a parahexus by proving any *one* of the following:

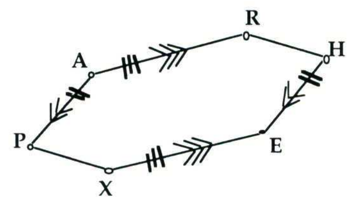
All pairs of opposite sides are parallel and one pair of opposite sides is congruent.



All pairs of opposite sides are congruent and one pair of opposite sides is parallel.

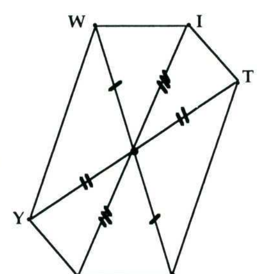


Two pairs of opposite sides are both parallel and congruent.

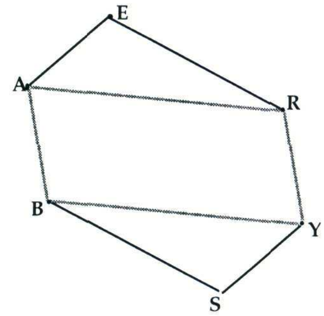


In all three cases the remaining parallels or congruences can be verified.

Theorem 1: Other ways to prove that a hexagon is a parahexus are also possible. If the 3 main diagonals of a hexagon are concurrent, and the point of concurrence is the midpoint of each diagonal, then the hexagon is a parahexus.

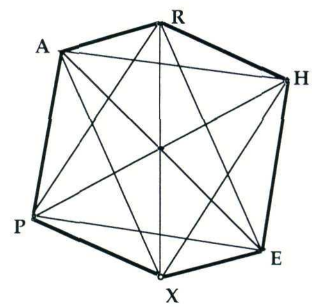


Justification: Using SAS and the fact that vertical angles are congruent you can prove the triangles opposite each other across the point of concurrence are congruent and thus prove opposite sides congruent, and prove them parallel using the congruent alternate interior angles. This is very much like the "diagonals bisect each other" method for proving that a quadrilateral is a parallelogram.



Theorem 2: A hexagon formed by a parallelogram with congruent triangles placed on 1 pair of opposite sides of the parallelogram so that opposite sides of the resulting hexagon are congruent forms a parahexus.

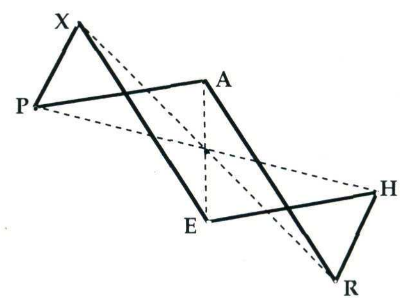
Justification: One pair of opposite sides are already parallel and congruent, and the other 2 pairs are congruent because they are corresponding parts of congruent triangles.



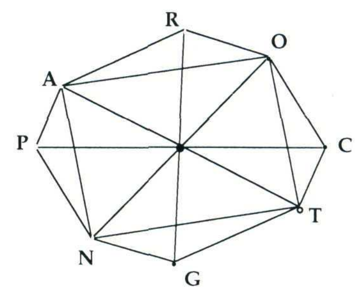
Properties of a parahexus:

- Each pair of opposite angles are congruent.
- Any pair of diagonals that do not intersect inside or on the parahexus are parallel.
- The diagonals connecting the endpoints of opposite sides, along with the sides themselves, form a parallelogram.
- A triangle whose vertices are consecutive vertices of a parahexus is congruent to the triangle determined by the three remaining points of the parahexus.
- The three main diagonals bisect one another.
- Using the congruent triangles, one can prove that any one main diagonal bisects any other main diagonal. A corollary is that the diagonals of a parahexes are concurrent.
- Triangles determined by alternating vertices are congruent.

These properties can be verified using the parallel and congruent sides to establish many congruent triangles. Be sure to observe the many parallelograms formed by the sides and diagonals of the parahexus.

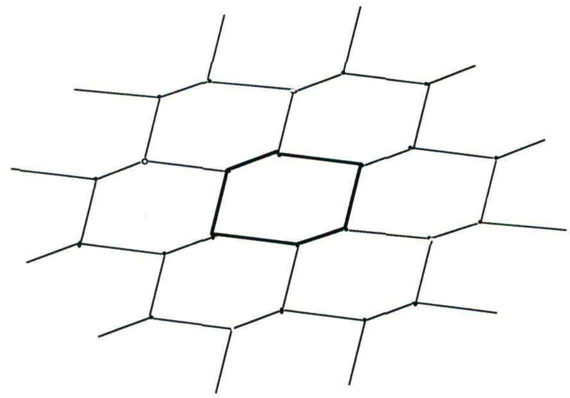


If the parahexus is warped so that it is no longer a convex figure, some diagonals will go outside the figure, but all the properties will hold.

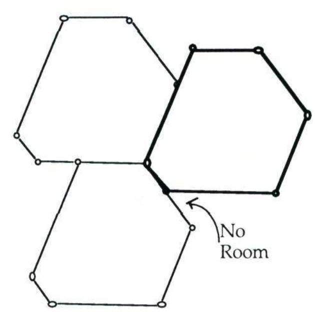


Most of the parahexus properties can be extended to further polygons. Any n-gon in which n is even can be a *parallelopolygon*. In all cases, you can prove congruent triangles are formed by the sides and the main diagonals and that the main diagonals both bisect each other and are concurrent.

Parahexes do have a useful property. (What? These things can actually be used for something?) Any parahexus can be used to tessellate a plane. Given any parahexus, one can place six identical ones, oriented in the same direction, on each side of it. These will match up perfectly because pairs of opposite sides are congruent the 3 angles around the vertex where the parahexes meet will add up perfectly to 360°. This is a consequence of the fact that opposite angles are congruent and the sum of the angles is 720°. More parahexes can be added to the edges of these, and more to those, on to infinity. Interesting shaped parahexes could be used as floor tiles, being almost as easy to make and a lot better looking than plain old squares or regular hexagons.



Also, it is appears that no hexagon <u>except</u> a parahexus can infinitely tessellate a plane by simple translation. If pairs of opposite sides of a given hexagon are not all congruent, then the sides would not fit together and gaps would be left. This is rather obvious. And if opposite sides are not parallel, three adjacent angles (which would be the same as the three angles around a point where 3 hexagons meet) would not add up to 360°. This means all opposite sides must be parallel and congruent, therefore, a parahexus is necessary. Concave parahexes will also work for tessellating a plane.



This feature of tessellating a plane does not extend to parallelo-8-gons, //-10-gons, //-12-gons, //-50-gons, etc. ✍

Additional Questions:

1) Find some other concise methods to prove that a hexagon is a parahexus?
2) Prove that a figure formed by connecting alternate vertices of a paraoctagon is a parallelogram?
3) What special features does a paradecagon have?
4) What minimal conditions are necessary for two parahexes to be congruent?
5) What minimal conditions are necessary for two parahexes to be similar?

PERCENT INCREASES AND DECREASES: WHAT DO THEY MEAN?

by: Charles L. Hamberg and Susan K. Eddins
Illinois Mathematics and Science Academy

Consider the following problem:

The value of each share of HiTech stock, when issued in 1993, was \$20. During 1993 the stock increased 20% in value. In 1994 the stock's annual performance showed a decrease of 10%. In 1995, each share showed an annual gain of 40%. What is the current value of the stock?

Students' who have little or no experience with or understanding of the stock market and changes in stock value might easily come up with two interpretations for this problem.

Interpretation #1

		Value
Initial value of stock when issued in 1993		\$20.00
20% increase in 1993	(.20 x \$20 = \$4)	\$24.00
10% decrease in 1994	(.10 x \$24 = \$2.40)	\$21.60
40% increase in 1995	(.40 x \$21.60 = \$8.64)	\$30.24

The percentage increase in the stock from its initial date of issue in 1993 to the end of 1995 can be calculated as $\frac{\text{Final Value - Initial Value}}{\text{Initial Value}}$.

Using Interpretation #1 this would be $\frac{30.24 - 20.00}{20.00} = \frac{10.24}{20.00} = .512$ or 51.2%.

Interpretation #2

		Value
Initial value of stock when issued in 1993		\$20.00
20% increase in 1993	(.20 x \$20) = \$4	\$24.00
10% decrease in 1994	(.10 x \$20) = \$2	\$22.00
40% increase in 1995	(.40 x \$20) = \$8	\$30.00

This interpretation assumes that the percentage increase is always figured on the initial value. The percentage increase in the stock from its initial date of issue in 1993 to the end of 1995 according to Interpretation #2 would be $\frac{30 - 20}{20} = \frac{10}{20} = 50\%$.

Question: What was the "average" annual increase for 1993, 1994, and 1995? Is it approximately 16.7% which might be calculated from $50\% \div 3 \approx 16.7\%$ or the average of the annual percentages $\frac{20\% - 10\% + 40\%}{3} \approx 16.7\%$? Or is it $51.2\% \div 3 \approx 17.1\%$?

When dealing with stocks, interpretation #1 is the more common way of viewing the percent changes in the value. That is, percentage gain is "benchmarked" against the previous year's closing price.

Interpretation #2 is a common way of interpreting the change in purchasing power for a fixed amount of money in a given year. Thus, a 5% increase in inflation is often interpreted to mean a 5% decrease in purchasing power as indexed to a fixed year.

An understanding of the different interpretations is important for all citizens since data is often reported to consumers without explicitly clarifying which method is being used for calculations. The table below describes the audited performance record of Wall Street expert John Dessauer's recommended portfolio for stocks. It leaves room for multiple investment interpretation.

	Annual Performance	Cumulative Gain
1982	20.1%	20.1%
1983	21%	45.3%
1984	0.0%	45.3%
1985	41.8%	106.1%
1986	20.2%	147.7%
1987	–10.9%	120.7%
1988	27.7%	181.8%
1989	21.1%	241.3%
1990	–16.5%	185.0%
1991	36.8%	289.8%
1992	11.8%	335.8%
1993	23.,1%	436.5%
1994	–5.8%	405.4%

(A hypothetical investment of $150,000 in 1982 would have been worth $758,100 in 1994.)

Look at this table carefully.

- How is it possible for an annual gain in 1988 of 27.7% to translate into a cumulative gain of 61.1%?
- How is it possible for an annual gain in 1993 of 23.1% to translate into a cumulative gain of 100.7%?
- How can an annual gain which is less in 1993 than in 1988 can produce a larger percent cumulative gain in 1993 then in 1988?
- If the average annual rate of inflation during this time period was 4%, what gain in *purchasing power* does this investment represent?

The preceding illustrates how data can give a distorted impression depending upon the interpretations and assumptions given to the data. Being aware of such interpretations and assumptions empowers each of us to have a greater understanding of the "real world" and to be more intelligent consumers in this "Information Age". ✍

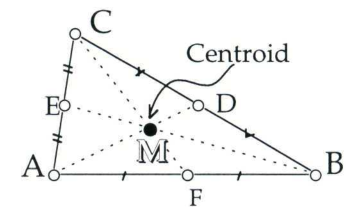
TRIANGLE CENTERS: Points of Concurrency and Symmetric Points

by: George Milauskas
Illinois Mathematics and Science Academy

It is typical in a Geometry class to ask students to locate four special points of a triangle.

(1) The Centroid

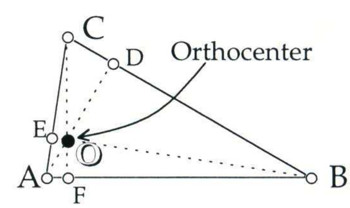
The *Centroid*, $\mathbb{M}$, of a triangle is the point of concurrency of the medians of the triangle, and has the following properties:



- The areas of all six sub-triangles are equal.
- The centroid is the center of mass of the triangle.
  (That is, the triangle will balance at its centroid, as well as balance along any of its medians.)
- It divides each median in a ratio, 2:1. (ex: CM:MF = 2:1)

(2) The Orthocenter

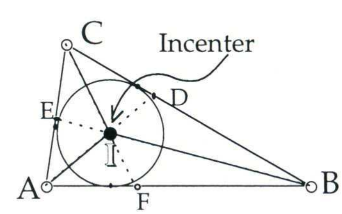
The *Orthocenter*, $\mathbb{O}$, of a triangle is the point of concurrency of the altitudes of the triangle, and has the following properties:



- Several pairs of similar triangles are formed.
  (ex. $\Delta CFB \sim \Delta ADB$, $\Delta AFC \sim \Delta AEB$, $\Delta CDA \sim \Delta CEB$)
- The quadrilaterals FODB, FOEA, DOEC are cyclic.
  (Since opposite angles are supplementary, each quadrilateral is inscribable in a circle)

(3) The Incenter

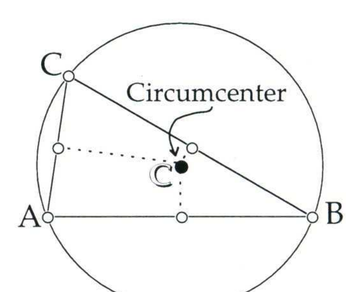
The *Incenter*, $\mathbb{I}$, of a triangle is the point of concurrency of the angle bisectors of the triangle. Some of the properties we see are:



- It is the center of an inscribed circle.
  (radius $r = \frac{2 \cdot \textbf{Area}}{\text{Perimeter}}$)
- The area of the triangle is numerically equal to the perimeter, whenever r = 2.
- $\mathbb{I}$ is equidistant from the sides of the triangle.

(4) The Circumcenter

The *Circumcenter* , $\mathbb{C}$, of a triangle is the point of concurrency of the perpendicular bisectors of the sides of the triangle. Some properties we see are:



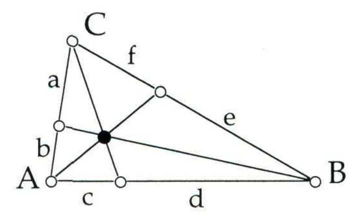
- It is the center of the circumscribed circle.
  (radius $R = \frac{a \bullet b \bullet c}{4k}$ where **k** is the area of Δ)
- $\mathbb{C}$ is equidistant from the vertices of the triangle.

Activity: (By ruler and compass construction or by using computer construction software)[1]

Determine the relative locations of the 4 "centers" in triangles which are:
(a) Acute (b) Right (c) Obtuse (d) Isosceles (e) Equilateral

Ceva's Theorem:

A neat little theorem which extends beyond these four special points, Ceva's Theorem, involves points of concurrency of cevians. A *cevian* is a segment which joins a vertex of a triangle to any interior point of the opposite side. Notice that the median is an example of a *cevian*. Ceva's Theorem indicates that three cevians of a triangle will be concurrent when and only when the product of the ratios of consecutive side segments is **1**.



That is: $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = 1 \iff$ the cevians are concurrent.

The proof of Ceva's Theorem is found in many books, such as (2). *You might want to try this proof on your own. It isn't too difficult.*

Problems: Use Ceva's Theorem

1. Prove that the medians of a triangle are concurrent.
2. Prove that the three angle bisectors of a triangle are concurrent.
3. Prove that the three altitudes of an acute triangle are concurrent.

Mass Points:

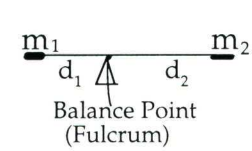
Cevians lead to a very interesting mathematical application of a physical principle.

That is, balance occurs whenever the ratio of masses is inversely proportional to the ratio of distances to the balance point.

For any balance beam: $m_1 \cdot d_1 = m_2 \cdot d_2$ or $\frac{m_1}{m_2} = \frac{d_2}{d_1}$

Note also that the total mass at the fulcrum is: $m_1 + m_2$

The Fulcrum Problem:



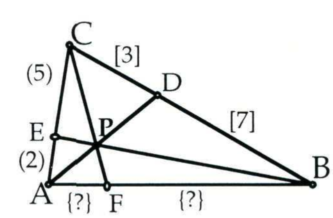
Mass Points applied to a triangle:

An extension of the fulcrum problem can be applied to triangles.

Given ΔABC with AE:EC = 2:5, CD:DB = 3:7

(Ratios in diagram are represented as numbers within brackets)

We can find AF:FB and also find AP:PF, AP:PD, EP:PB. by thinking of these numbers as distances and looking at the masses needed to "balance" the system.

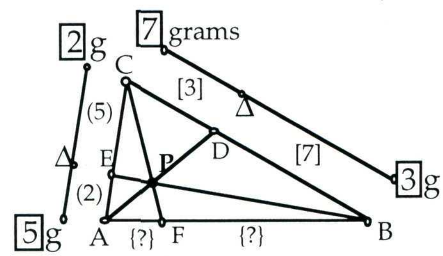


We can make the side "balance" at the interior point by placing appropriate masses at the ends.

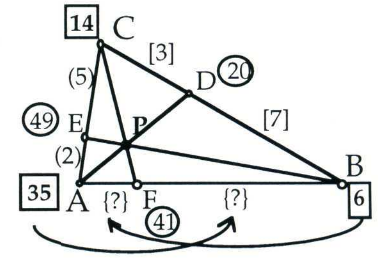
$\overline{CB}$ balances at D with masses of 7 g and 3 g at the ends.

$\overline{AC}$ will balance at E with masses of 5 g at A and 2 g at C.

However, in order to accomodate both the 2 g and 7 g needed at C, we can increase all the masses proportionally to make a mass of 14 grams at C.



In physics, when a fulcrum is placed so that the product mass x distance (mass to fulcrum) is the same on both sides of the fulcrum the system will balance.



Thus, we have masses of 35, 14, and 6 at the vertices, and total masses of: 49, 20, and 41, at the balance points on the sides. It is evident in the figure that AF:FB = 6:35 (here we know the "masses" we need to determine the "distances".) Similarly, the cevians are divided such that, AP:PD = 20:35, CP:PF = 14:41, and EP:PB = 6:49.

Take note that along any cevian, the point P is a balance point with total mass of $49 + 6 = 35 + 20 = 14 + 41 = 55$ which is the sum of the original three masses added at the vertices. (14+35+6)

With the masses in place, ΔABC will balance at P.

Mass Points provides a way to obtain ratios of side segments and of the parts of cevian segments.

Now return to Triangle Centers.
Each Central Point has a "symmetry" within the triangle.

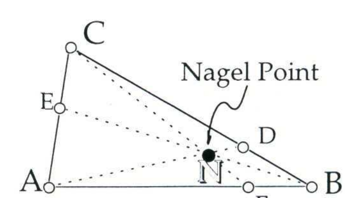
- The medians and centroid are said to "balance area". The centroid is a central point relative to area. See if you can use mass points to justify that the centroid divides each median in a ratio 2:1.
- The incenter is "equidistant from the sides" of the triangle. It is a central point relative to the sides. Using Ceva's theorem, we can prove that the angle bisectors are concurrent. Since an angle bisector divides the opposite side of the triangle in the same ratio as the sides of the angle, $\frac{AC}{AB}=\frac{CD}{DB}$ and $\frac{CB}{AC}=\frac{FB}{AF}$ and $\frac{AB}{CB}=\frac{AE}{EC}$ Therefore: $\frac{CD}{DB}\cdot\frac{FB}{AF}\cdot\frac{AE}{EC} = \frac{AC}{AB}\cdot\frac{CB}{AC}\cdot\frac{AB}{CB} = 1$
- The circumcenter is "equidistant"from the vertices" of the triangle and is a central point relative to the vertices. Even though it was not initially determined by three cevians, the circumcenter has a unique threesome of cevians which can be drawn through it. Ceva's theorem guarantees that the product of the ratios of the parts of each side will be 1. It is not an easy task to determine these individual ratios nor to demonstrate that the product is 1.

Other Central Points In Every Triangle:

The following are additional "centers" in a triangle.

(5) The Nagel Point

The *Nagel Point* , $\mathbb{N}$, of a triangle is the point of concurrency of the segments that join each vertex to the "semi-perimeter" point, the point halfway around the perimeter of the triangle from the vertex.

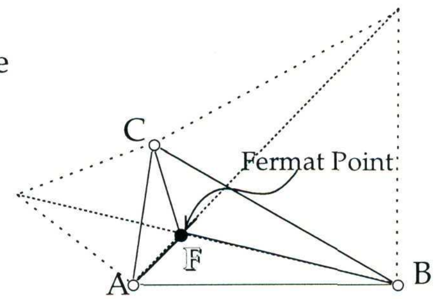


- You can show that: CE=BF, AE=DB, and AF=CD. Thus, vertical angles at the Nagel Point cut off equal segments on the sides.
- Use Cevian's Theorem to prove that the 3 semi-perimeter cevians are indeed concurrent.

(6) The Fermat Point

The *Fermat Point* , $\mathbb{F}$, of a triangle is the point where the sum of the distances to the vertices, CF+BF+AF, is a minimum.

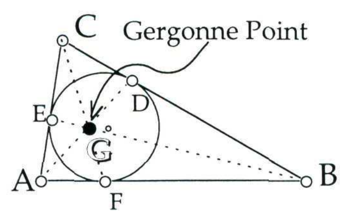
- The angles CFB, BFA and AFC are equal, 120°.
- Each cevian can be constructed by joining the outer vertex of an equilateral triangle mounted on a side to the opposite vertex of the triangle.



(7) The Gergonne Point

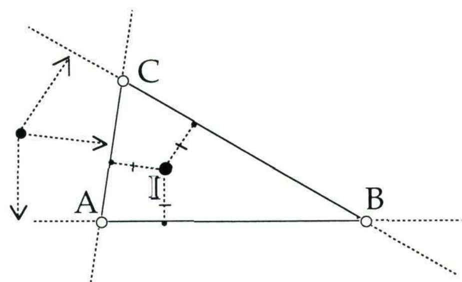
The *Gergonne Point* , $\mathbb{G}$, of a triangle is the point of concurrency of the segments that join each vertex to the opposite point of contact of the circle with the side of the triangle.

- Use the "two tangent theorem" to prove that the Gergonne Point exists. That is, that the three cevians determined by the points of tangency are concurrent.



Additional Problems:

1) Find 3 other points besides the incenter that are equidistant from the lines that contain the sides of the triangle. (i.e. the extensions of the triangle.) Join them to the opposite vertices to obtain another central point. One such point is shown at the right.



2) Justify why every point inside a triangle is uniquely determined by a threesome of cevians.

These are only a few of the hundreds of such symmetric points that have been studied relative to triangles. You might try discovering a central point that you can name. Try to prove the concurrency of your defined cevians and try to find some interesting properties. ✍

References:

(1) Dynamic Geometric Construction Software (available on both PC & Mac platforms)
- *Cabri* Geometry II, available through Texas Instruments Company
- *Geometer's Sketchpad*, Key Curriculum Press, (800) 338-7638

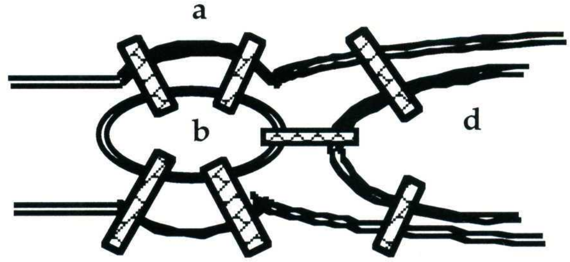
(2) *Excursions In Advanced Geometry*, Posamentier, Addison-Wesley Publishing Co., Inc., 1 Jacobs Way, Reading, MA 01867, (800) 552-2259.

(3) "Central Points and Central Lines in the Plane of a Triangle", an article by Clark Kimberling, University of Evansville, Evansville, IN 47722

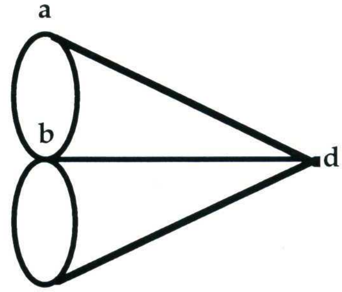
MATHEMATIZATION: A WALK IN THE PARK

by: Susan K. Eddins
Illinois Mathematics and Science Academy

There is a park, *Jardin de la Ville Amélia,* in Barcelona, Spain, where most mornings the older men of the neighborhood meet to walk and visit. While walking in this park with a Spanish colleague during a conference this past fall, I was reminded of Euler's famous *Königsberg Bridge Problem* of 1736 which is often cited as the starting point for the mathematical field know as *graph theory.*



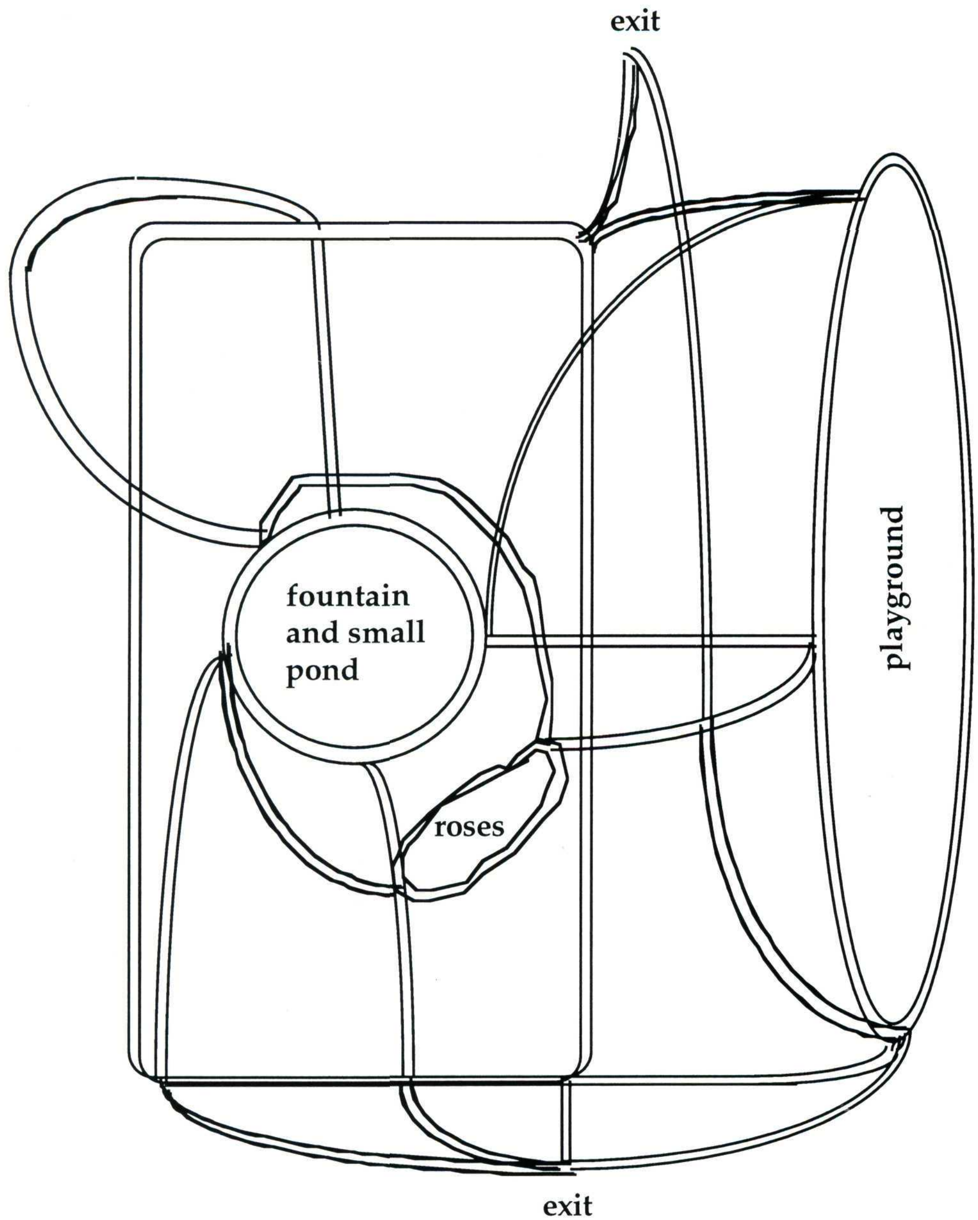
The German town of Königsberg was built on both sides of a river. The town included two islands connected to the shores and each other by a series of seven bridges. The question posed by Euler was whether residents of Königsberg could take a stroll in the evening during which they would cross each of the seven bridges exactly one time. To answer this question, Euler *mathematized* it, that is he took the *essential elements* of the situation and represented them using *mathematical objects.* In this case he used segments of lines and curves and their points of intersection.



Euler chose to represent each land mass by a point, called a *vertex,* and each bridge by a segment, called an *edge.* When mathematized in this way, Euler's question is the same as asking whether it is possible to trace over every segment of this *network* exactly once without lifting your pencil.

Euler reasoned as follows: If the *degree* of a vertex is odd, that is, if it has an odd number of edges coming into it, then you must either *begin* or *end* your tracing at this vertex. To approach a vertex along one edge and leave along another requires a *pair* of edges. Because all four vertices in the Königsberg network are of *odd degree,* traveling across each bridge exactly one time is **not** possible.

The challenge presented to you is to use the drawing of the *Jardin de la Ville Amélia* to mathematize the situation and to determine whether Dr. Perez-Pardo could walk along every path in the park exactly one time during his morning stroll. If it is possible, where must he begin and end? If it is not possible in one walk, how many different walks would it take for him to be able to do so and where could he begin and end each? [Mail you solutions along with your comments back to IMSA.]



Jardin de la Ville Amélia

[Hint: You may need to think carefully about what you represent as vertices.] ✍

• Kenney, Margaret J. (editor), Discrete Mathematics Across the Curriculum, (the 1991 Yearbook of the National Council of Teachers of Mathematics), NCTM, Reston, VA, 1991.

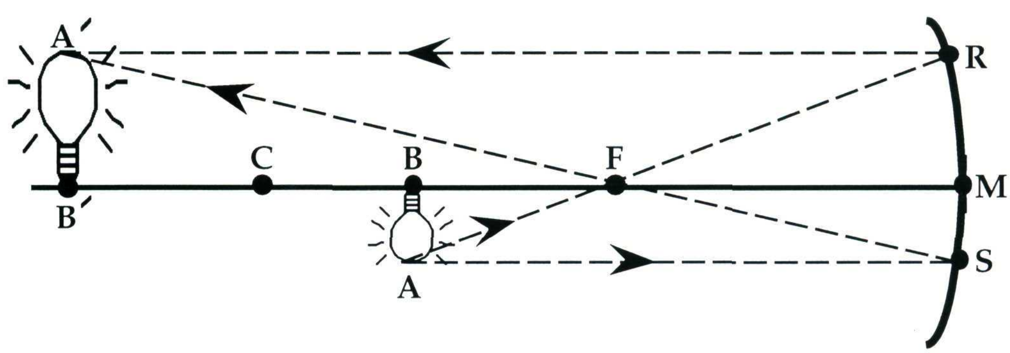
THE GEOMETRY OF REAL IMAGES

by: Dr. Raymond J. Dagenais
Illinois Mathematics and Science Academy

Mathematical modeling is a useful technique for describing phenomena or predicting results. Such techniques can be employed in the study of *real image* formation by concave mirrors. Using some fundamental geometric concepts and taking into account the physical properties of the mirror it is possible to model the production of a *real image* and make some predictions about the image that is formed.

In the diagram below, AB represents an object stationed at a distance d_o in front of a concave mirror whose surface is represented by the curve RMS. A'B' is the *real image* of object AB, that is, it is the location where light rays from the object reflected from the mirror converge. (A virtual image is the location where light rays coming from the object appear to converge "behind" the mirror.) In the situation pictured, the image is located outside of C, which is the center of curvature of the mirror at a distance d_i from the mirror's surface. Incident light rays coming from the object that are parallel to the principal axis MC will reflect off the mirror's surface such that they will always pass through the focal point, F. Incident light rays coming off the object that pass through F will reflect back parallel to the principal axis.

The law of reflection that is important in this discussion states that the angle of incidence of an incoming light ray is equal to the angle of reflection of an outgoing light ray. The angle of incidence is defined to be the angle between the incoming light ray and the normal (or perpendicular) to the surface at that point. The angle of reflection is the angle between the reflected light ray and the normal.



The following definitions help to predict some important characteristics of the *real image*.

$$MF = f \qquad MB = d_o \qquad MB' = d_i$$

In spherically shaped concave mirrors MF = $\left(\frac{1}{2}\right)$MC.

Using the diagram, it can be shown that:

$$\Delta\ ABM \sim \Delta\ A'B'M$$

[Draw in segments AM and A'M. $m\angle ABM = m\angle A'B'M = 90°$ and the measure of incident $\angle AMB$ equals the measure of reflection $\angle A'MB$.]

$$\Delta\ RMF \sim \Delta\ ABF$$

[Draw in segment RM. $m\angle ABM = m\angle RMF = 90°$ and $\angle BFA \approx \angle RFM$ because they are vertical angles.]

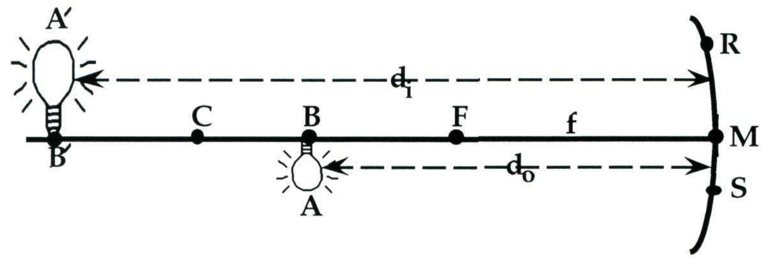
Employing the properties of similar triangles:

$$\frac{AB}{A'B'} = \frac{MB}{MB'} \quad \text{and} \quad \frac{AB}{RM} = \frac{BF}{MF}.$$

Since A'R is parallel to B'M, RM = A'B',

And it is clear that BF = BM – MF.

Putting this all together we find:



$$\frac{AB}{A'B'} = \frac{d_o}{d_i} \quad \text{and} \quad \frac{AB}{A'B'} = \frac{AB}{RM} = \frac{BF}{MF} = \frac{(BM - MF)}{MF} = \frac{(d_o - f)}{f}$$

So,

$$\frac{d_o}{d_i} = \frac{(d_o - f)}{f}$$

$$d_i = \frac{fd_o}{(d_o - f)}$$

This equation describes the image distance, d_i, in terms of the object distance, d_o, and the focal length, f, of the mirror. It models an important physical result.

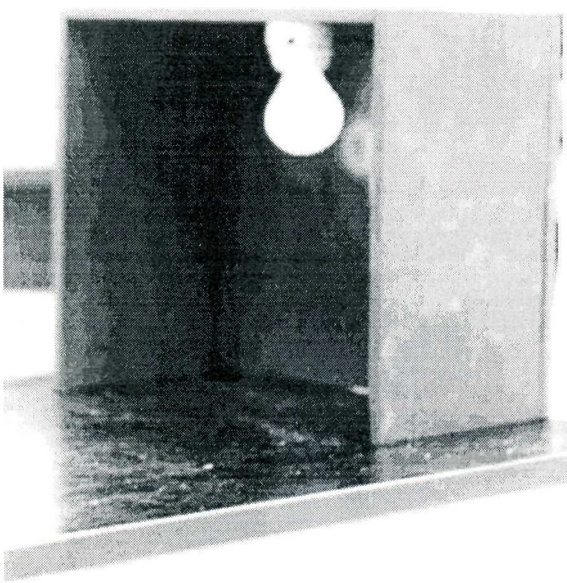
An interesting case to consider is the limit of the image distance as the object distance approaches MC, that is as d_o approaches 2f.

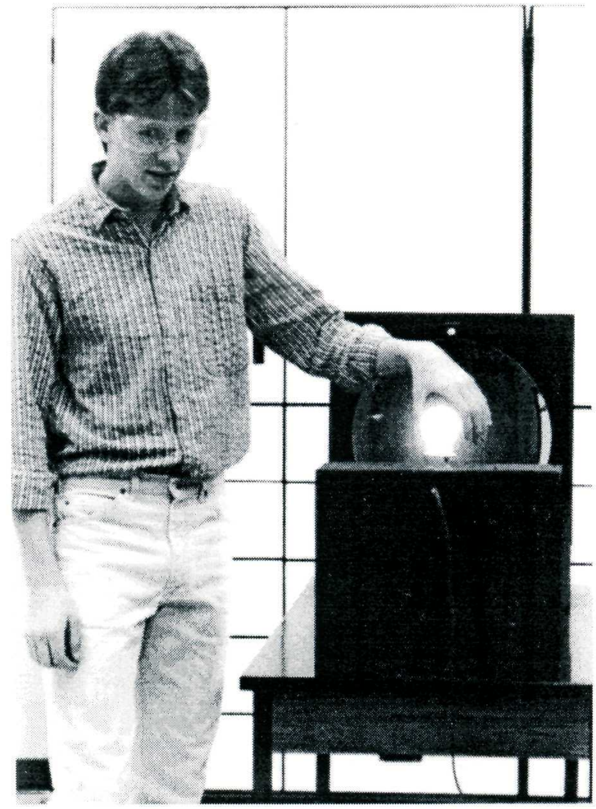
What is the $\lim\limits_{d_o \to 2f} d_i$?

What does this limit say about the location of the image?

Using similar triangles show how the height of the image compares to the height of the object as d_o->MC.

The photographs that follow show the object (inverted light bulb inside the box) and the real image (upright "light bulb" on top of the box). The limit calculations correctly predict the position and the height of the image formed by the concave mirror for the case where the object is located at C. ✍





WHAT'S WRONG WITH THIS PROOF?

by: Susan K. Eddins
Illinois Mathematics and Science Academy

If $a = b = 1$,

then
$$a^2 = a \cdot b$$
$$a^2 - b^2 = a \cdot b - b^2$$
$$(a + b)(a - b) = b(a - b)$$

So
$$a + b = b$$
$$1 + 1 = 1$$

2 = 1 **(SURPRISE!)** ✍

SOME EXPLORATIONS WITH EVEN AND ODD FUNCTIONS

by: Charles L. Hamberg
Illinois Mathematics and Science Academy

Mathematics textbooks often include a brief discussion of odd and even functions, particularly as they relate to symmetries of graphs about the x-axis and the origin. In this article we shall consider several exercises which deal with odd and even functions not only with regard to their graphs but also by asking students to carefully use definitions, form conjectures, give convincing counter examples, or validate their conclusions. In our work we will limit our discussion to odd and even functions of x.

Exercises

Definition: A function f is even if $f(x) = f(-x)$ for all values of x in the domain of f.

1. Explain why the following functions are even functions:

   (a) $f(x) = |\,x\,|$ (b) $g(x) = \cos x$

   (c) $h(x) = 2^x + 2^{-x}$ (d) $j(x) = \begin{cases} y = t^6 \\ x = t^3 \end{cases}$

   (e) $k(x) = 6$ (f) $l(x) = x^4 - x^2 + 3$

2. Explain why a function f is even if $f(x) - f(-x) = 0$ for all values of x in the domain of f.

3. Explain why $f(x) = x^2$ is not an even function for $x \in [-2,4]$.

4. Explain why $f(x) = \dfrac{x^4 - x^2 + 6}{(x-3)(x+1)}$ is not an even function.

Definition: A function f is an odd function if $f(x) = -f(-x)$ for all values of x in the domain of f. Alternatively, if $f(x) + f(-x) = 0$ for all values of x in the domain of f.

5. Describe which of the following functions are even functions, which are odd functions and which are neither.

   (a) $f(x) = x^3$ (b) $g(x) = \log|\,x\,|$ c) $h(x) = 2^x - 2^{-x}$

   (d) $j(x) = \begin{cases} y = t^9 \\ x = t^3 \end{cases}$ (e) $k(x) = x^7 - x^3 + x$ f) $l(x) = (x+3)^2$

6. Explain why $y = \dfrac{x(x-2)(x+2)}{(x+1)(x-1)}$ is an odd function.

7. In the study of calculus, we will see that,
$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots + \dfrac{(-1)^{n+1}x^{2n-2}}{(2n-2)!} + \ldots$ for all x
and $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots + \dfrac{(-1)^{n+1}x^{2n-1}}{(2n-1)!} + \ldots$. Using these expressions, explain why $\cos x$ is even and $\sin x$ is odd.

Some Further Exercises

1. Explain why the sum of two even functions is an even function and the sum of two odd fuctions is an odd function.

2. (a) Given an example to show that the sum of 2 even functions can be **both** even and odd.

   (b) Give an example to show that the sum of 2 odd functions can be **both** even and odd.

3. Explain why the product of two odd functions will **always** be an even function.

4. (a) $(a,f(a))$, $a \neq 0$, is on the graph of even function $y = f(x)$. What other point is guaranteed to be on the graph?

   (b) $(a,g(a))$, $a \neq 0$, is on the graph of odd function $y = g(x)$. What 2 points are guaranteed to be in the graph?

5. Explain how it is possible for an infinite number of functions to be **both** odd and even.

6. Give an example to show that the sum of 2 non-odd functions can be odd.

7. Classify each of the six trigonometry functions as being either "odd" or "even".

8. Explain why $y = \log x^2$ is even but $y = 2 \log x$ is not.

9. Classify each of these principal-valued inverse trigonometric functions as "odd", "even", or "neither", or "both"

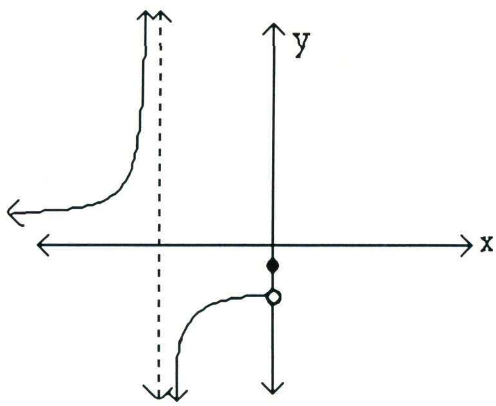
(a) $y = \sin^{-1}x$ (b) $y = \cos^{-1}x$ (c) $y = \tan^{-1}x$

10. Write definitions so that one can determine if $f \circ g$ is an even or odd function.

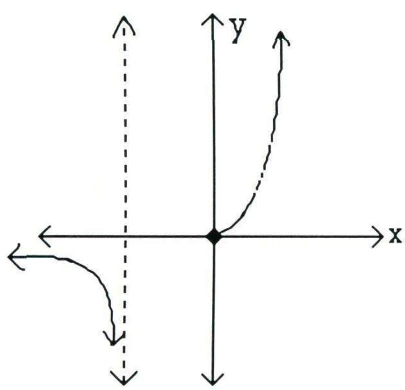
11. Let f be an even function and g be an odd function. Classify as "odd" or "even".

(a) $f \circ f$ (c) $f \circ g$

(b) $g \circ g$ (d) $g \circ f$

12. 

Given: f is an even function. Complete the graph.

13. 

Given: f is an odd function. Complete the graph.

14. What restrictions on the domain of f would enable $f(x) = x^5 - 4x^3$ to be an even function? ✍

CONNECTIONS INVOLVING LINEAR AND QUADRATIC FUNCTIONS

by: George Milauakas and Charles L. Hamberg
Illinois Mathematics and Science Academy

The study of linear and quadratic functions traditionally occupies a significant amount of the elementary and intermediate algebra curricula. As mathematics instructors, we often limit our students' contact with these concepts by restricting ourselves to traditional problems and exercises. Here we offer several types of problems which relate to linear and quadratic functions. It is hoped that these examples will spur the readers to contemplate, consider, and incorporate similar types of connections and extensions into their classes.

1) Determine the range of the function $f(x) = 25 - x^2$ as $-3 \le x \le 4$.

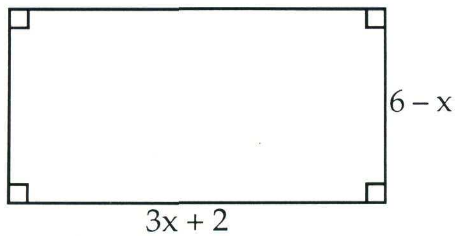
2) The motion and path of a particle is determined by $x(t) = t^2$ and $y(t) = 5 - t^2$. (Now that the graphing calculator is available, working with parametric equations becomes accessible even to beginning algebra students.)
   (a) Represent the motion and path of the particle in the coordinate plane for $-2 \le t \le 5$.
   (b) Is this graph a piece of a linear or quadratic function? Justify your answer.
   (c) Determine the distance traveled by the particle in part (a).
   (d) If t can take on *any* real value, draw the resulting graph?

3) For $x > 0$, let $P(x) = 1 + \frac{6}{x}$. Find the value of the continued fraction represented by $P \circ P \circ P \circ P \circ \ldots(x)$. Remember, $x > 0$ so $P(x) > 0$.

   Hint: $P \circ P \circ P \circ P \circ \ldots(x) =$

$$1 + \cfrac{6}{1 + \cfrac{6}{1 + \cfrac{6}{1 + \cfrac{6}{\cdots}}}}$$

4) Let $f(x) = x^2 - 5x - 14$.
   (a) Determine all x-values so that $f(x) = 0$.
   (b) Use your results from part (a) to determine all x-values so that $f(2x - 6) = 0$. [Remember we could let $u = 2x - 6$ and part (a) gives us the value of u when $f(u) = 0$.]
   (c) Draw the graphs of $y = f(x)$ and $y = f(2x - 6)$ on the same set of axes. Think about how the geometric transformations of the graphs relate to the algebraic tranformations performed in part (b).

5) 

(a) What restrictions need to be placed on x so that this rectangle exists?

(b) Write a function $y = A(x)$ which describes the area of the rectangle shown.

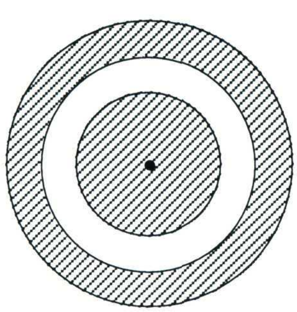
(c) Find all values of x so that $A(x) = 2A(2)$. Write a sentence which gives an interpretation to this equation.

6) Draw the graphs for $f(x) = x^2 + 3$ and $g(x) = -x^2 + 4x - 5$ on the same set of axes. Define $y = d(x)$ to be the vertical distance between f and g for each x. Determine the value of x so that the function d is minimized.

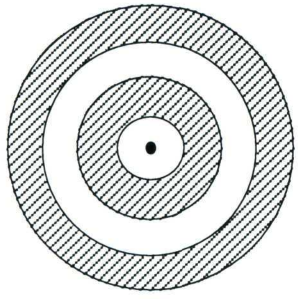
7) Let $g(x,y) = 3x - 2y - 17$. Determine the area of the triangle formed by the lines $x = 0, y = 0$ and $g(x + 3, y - 1) = 0$.

8) Let $g(x,y,r) = x^2 + y^2 - r^2$. Determine the area of the region described by $g(x - 2, y + 3, 7) \leq 14$.

9) Determine all **x** values where: $k(\mathbf{x}) = \mathbf{x}^2 - \mathbf{x} - 2$ and $k(k(\mathbf{x})) = 0$.

10) 

The radii of the 3 concentric circles shown are x, $x + 2$, and $x + 4$. Draw the graph of all points $(x, A(x))$ where $A(x)$ is the shaded area shown.

11) 

The radii of the 4 concentric circles shown are x, $x + 2$, $x + 4$, and $x + 6$. Draw the graph of all points $(x, A(x))$ where $A(x)$ is the shaded area shown.

CONTINUED FRACTIONS AND COMPOSITION FUNCTIONS

by: Ben Chelf, Class of 1996
Illinois Mathematics and Science Academy

In problem 3 from "Connections Involving Linear and Quadratic Functions", we addressed the idea of a continued fraction generated by the function: $f(x)=1+\frac{6}{x}$, for $x > 0$. This type of problem is almost always presented with its restriction ($x > 0$), but we might raise the question

Why must x be greater than 0?

If we let $f(x)=1+\frac{6}{x}$, we quickly see that the problem we initially dealt with can be expressed as $\lim\limits_{n\to\infty} f^{[n]}(x)$ where,

$$f^{[n]}(x) = (f\circ f\circ f\ldots\circ f)(x).$$

However, we might want to consider what happens when n does *not* go to infinity.

Let $n = 3$ $$f^{[3]}(x)=1+\cfrac{6}{1+\cfrac{6}{1+\cfrac{6}{x}}}=\frac{13x+42}{7x+6}$$

The domain of this function includes all real numbers except for those which cause a 0 in a denominator at *any* point in the composition For $f^{[3]}$, these restrictions are:

$x=0$, -6, and $-\frac{6}{7}$. (Can you figure out why?)

Once the function $f^{[3]}$ has been simplified, it appeared as if we only need to restrict x from taking on the value $-\frac{6}{7}$. However, as we increase the value of n we would be unable to simplify the fraction if *any* denominator in the entire fraction takes on a zero value.

Now we have shown that for n=3, our function is well defined with a specific domain and this would be true for any finite value of n. However, when we extend the problem to create a continued fraction, we run into a problem. As n goes to infinity, x literally disappears. With an infinite number of compositions, we write $f^{[n]}(x)$ as the continued fraction $1+\cfrac{6}{1+\cfrac{6}{1+\cfrac{6}{1+\ldots}}}$ without any mention of x! When simplifying this continued fraction, we use a simple substitution to find its value:

$$\text{Let } y = 1+\cfrac{6}{1+\cfrac{6}{1+\cfrac{6}{1+\ldots}}}$$

$$y=1+\frac{6}{y}$$

Solving this we see that

$$(y-3)(y+2)=0$$

It would seem that there are two solutions $y = 3$ and $y = -2$.

Look back at the continued fraction, no negatives appear nor do any minus signs. Obviously, y will *not* have a negative value. However, with our function, x can still be negative thus possibly causing the infinite continued fraction to be negative. Intuitively, this is absurd, but yet it is something that must be considered. From our case of n=3, we see that we have 3 restrictions on x. As n goes to infinity, so does the number of restrictions on x. It can be shown that in every case the restrictions on x are negative rational numbers. Although we have an infinite number of negative rationals, we cannot assume that these are *all* the negative rationals. And we are still left with all the negative irrationals as possible values of x.

What is the problem? The problem is in the way we decided to solve the limit as n went to infinity. We arbitrarily "assigned" the continued fraction without realizing what it would do to the problem. By introducing the continued fraction, we have automatically restricted x to be positive just by the nature of the fraction. Although negative values can occur in the function, in the continued fraction, they cannot.

Since our continued fraction does not allow for negative values of x, we must be cautious when dealing with a negative solution. When we solved the quadratic, we got two values for the fraction, which is silly for the fraction, but makes sense for the function. For any x in the domain of f the limit will be equal to 3, except for -2. For x=-2, $f^{[n]}(x) = -2$ because the composition gets "stuck." Although the continued fractions ignored negative values of x, it still helped us to see how the function would behave over all of the reals. ✍

"IT'S REALLY VERY SIMPLE"

by: Sue Eddins and Kim Eddins
Illinois Mathematics and Science Academy
and Batavia High Shool

"Do you need any help with implicit differentiation," I asked my daughter, Kim. "No thanks, Mom, it's really pretty simple. The problem asks you to find four things, they give you three, you just have to look through the problem until you find them, then you solve for the fourth."

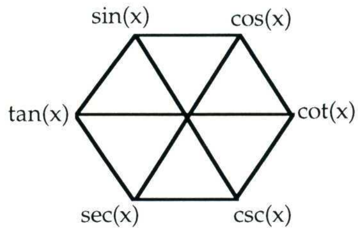
Kim has since amplified this simple little statement that seems to sum up about half of the math problems in the high school curriculum. She now knows that sometimes you need n things but they only give you n - 2 . Then you either have to look for two equations to solve or you first have to take the derivative, set it equal to zero, solve and substitute it back until you find all you need.

Guess it really is very simple when you look at it like that! ✍

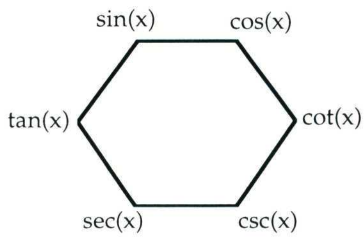
THE "JBTTMH"
(The JB Tate Trigonometric Mnemonic Hexagon)

by: Joe Tate
Illinois Mathematics and Science Academy

Label the vertices of a regular hexagon in **the following order** with the names of the six trigonometric functions. The *location* of each function can be used to help you remember many trigonometric identities.



1) All the main diagonals of this hexagon are connecting reciprocal functions.
e.g. $\sin(x) = \dfrac{1}{\csc(x)}$, etc . . .



2(a) The function at any vertex is equal to the ratio of the next two consecutive vertices in either direction.

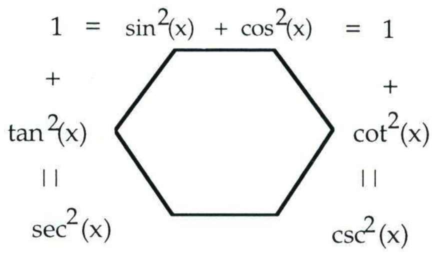
$$\text{e.g. } \sin(x) = \frac{\cos(x)}{\cot(x)}$$

$$\tan(x) = \frac{\sec(x)}{\csc(x)}$$

. . . and this works in both directions!

Another way to describe this relationship is:

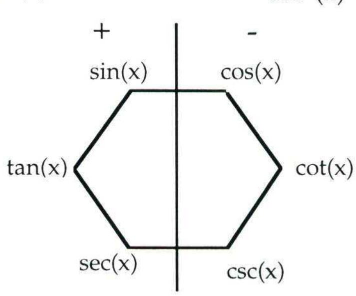
2(b) The function at any vertex is the product of the functions on either side. e.g. $\sec(x) = \tan(x) \bullet \csc(x)$.



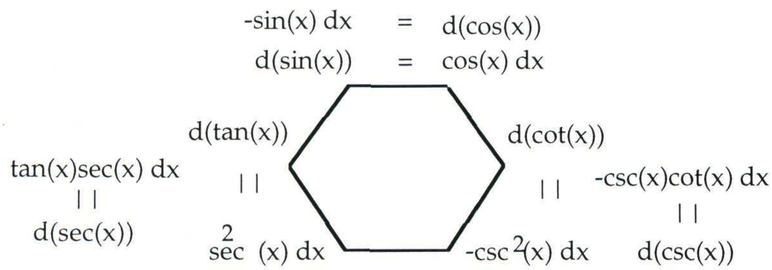
3) The familiar:

$\sin^2(x) + \cos^2(x) = 1$
$1 + \tan^2(x) = \sec^2(x)$
$1 + \cot^2(x) = \csc^2(x)$

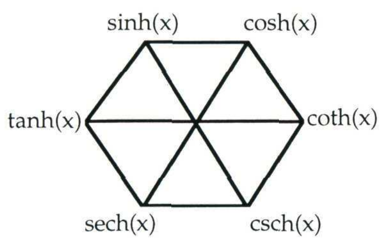
can also be found.



4) Patterns can also be found for the *derivatives* of the trigonometric functions. Sketch in a vertical line separating the hexagon into a left (positive) side and a right (negative) side. The derivative of any function on the left will be positive and the derivative of any function on the right will be negative. That is, the sign of the derivatives achieved using the functions on the left will not change. On the right there is a sign change implied by the negative in the formula.



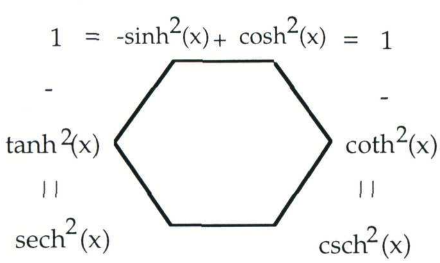
e.g. The derivatives of the sin(x) and the cos(x) are across from one another. The derivatives of tan(x) and cot(x) are "down" and squared. The derivatives of sec(x) and csc(x) "bounce" on sec(x) and csc(x) back up to tan(x) and cot(x).



5) This can be extended to the hyperbolic trigonometric functions.

a) The diagonals are reciprocals.

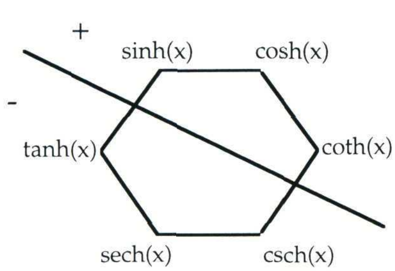
b) The function at any vertex is the product of those functions on either side.



c) $\cosh^2(x) - \sinh^2(x) = 1$

$1 - \tanh^2(x) = \text{sech}^2(x)$

$1 - \coth^2(x) = \text{csch}^2(x)$



d) Their derivatives also appear, but the line separating the hexagon into positive and negative halves is at an angle.

$$d(\sinh(x)) = \cosh(x)\,dx$$
$$d(\cosh(x)) = \sinh(x)\,dx$$
$$d(\coth(x)) = \text{csch}^2(x)\,dx$$
$$d(\tanh(x)) = -\text{sech}^2(x)\,dx$$
$$d(\text{sech}(x)) = -\text{sech}(x)\tanh(x)\,dx$$
$$d(\text{csch}(x)) = -\text{csch}(x)\coth(x)\,dx$$

If you find more relationships, please feel free to share them with us along with your comments on the IMSA Math Journal. We welcome your responses. (For further details please refer to the back side of the front cover for the mailing address.) ✍

THREE SPATIAL PYTHAGOREAN THEOREMS

by: Dr. Gregory Galperin
Eastern Illinois University

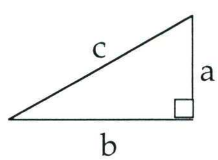
The aim of this article is to describe **three** distinct Pythagorean Theorems that hold in three dimensional space. I call them PT1, PT2, and PT3, where PT is the standard (plane) Pythagorean Theorem. In order to better understand the spatial pythagorean theorems, let us first revisit the plane Pythagorean Theorem.

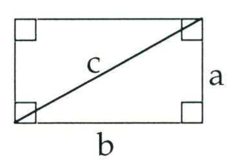
■ The Plane Pythagorean Theorem (PT) can be formulated in three different ways.
Each of these three ways results in the familiar equation, $c^2 = a^2 + b^2$.

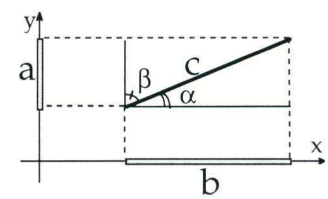
(i) The square of the hypotenuse of a right triangle equals the sum of the squares of the legs.

(ii) The square of the diagonal of a rectangle equals the sum of the squares of the sides.

(iii) The square of the length of a segment equals the sum of the squares of the segment's orthogonal projections onto any two mutually perpendicular lines. (**x** and **y** in the third figure)





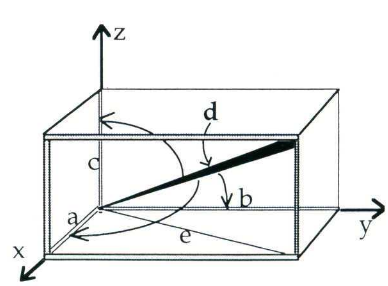


Denote by α and β the angles that a segment, **c**, in the plane makes with the perpendicular lines, **x** and **y**. The cosines of these angles are called the *direction cosines* of segment **c**. A very simple but important consequence of PT is:

♦ Direction Cosines Corollary
The sum of the squares of the direction cosines equals 1: $\cos^2\alpha + \cos^2\beta = 1$

[This is easily justified by considering that the projection of **c** onto the line **x** is: $b = c\cos\alpha$ or $\frac{b}{c} = \cos\alpha$ and the projection of **c** onto the line **y** is: $a = c\cos\beta$ or $\frac{a}{c} = \cos\beta$. Thus, you might recognize a fundamental formula of trigonometry, $\cos^2\alpha + \cos^2\beta = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{a^2+b^2}{c^2} = 1$]

■ The First Spatial Pythagorean Theorem (PT1)



(i) The square of the diagonal of a rectangular box equals the sum of the squares of its three perpendicular edges.

$$d^2 = a^2 + b^2 + c^2$$

(ii) The square of the length of a segment in space equals the sum of the squares of its projections onto three mutually perpendicular lines, X, Y, and Z.

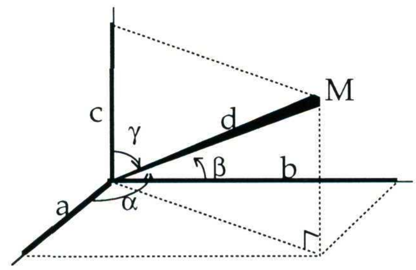
[Place the x-y-z-axes along the lines X, Y and Z, with one endpoint of the segment, **d**, to be at the origin. The segment is the diagonal of the box and the projections of the segment are now the edges of the box: $d^2 = e^2 + c^2 = (a^2 + b^2) + c^2$]

Let α, β, and γ, be the angles between a segment and three mutually perpendicular lines in space. The cosines of these angles are the *direction cosines.*

♦ Spatial Direction Cosines Corollary

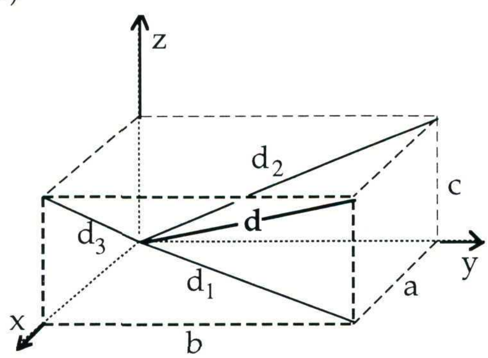
The sum of the squares of the direction cosines is always 1 and is independent of the position of the segment relative to the three mutually perpendicular lines:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$



The Second Spatial Pythagorean Theorem (PT2)

Let us project, orthogonally, a segment AB with length **d** onto three mutually perpendicular planes (which can be considered to be the xy, yz, and xz planes). Denote the lengths of the projections by: d_1, d_2, and d_3. [These are diagonals of the rectangular box we encountered earlier]



(i) The square of the length of a segment in space is half the sum of the squares of its projections onto three mutually perpendicular planes: $d^2 = \frac{1}{2}\left(d_1^{\,2} + d_2^{\,2} + d_3^{\,2}\right)$

[The proof of this theorem is based upon the fact that the projections of the segment AB onto the **x**, **y**, and **z** axes coincide with the projections of the segments $\mathbf{d_1}$, $\mathbf{d_2}$, and $\mathbf{d_3}$ onto the same axes. Denote these projections onto axes by **a**, **b**, and **c**.
From PT we have: $a^2 + b^2 = d_1^{\,2}$, $b^2 + c^2 = d_2^{\,2}$, and $c^2 + a^2 = d_3^{\,2}$
From PT1: $d^2 = a^2 + b^2 + c^2$. Therefore, $d_1^{\,2} + d_2^{\,2} + d_3^{\,2} = 2\cdot a^2 + 2\cdot b^2 + 2\cdot c^2 = 2d^2$.]

◊Problem 1 The sum of the surface areas of two cubes is equal to the surface area of a third cube. Which is greater: the sum of the volumes of the first two cubes or the volume of the third cube ?

◊Problem 2 Consider the three perpendicular projections of a segment of length one onto the x- y- and z-axes. Orient the segment so that the length, k, of its projection on the x-axis is less than or equal to the lengths of the y and z projections. Find the maximum possible value of k [Suppose the segment has one end at the origin, for how many positions of the segment in space will k attain its maximum value?]

◊Problem 3 (a) Evaluate the sum: $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

(b) Let $\alpha > \beta > \gamma$. Prove that $\cos\alpha < \frac{1}{\sqrt{3}}$, $\cos\beta < \frac{1}{\sqrt{2}}$ (i.e., $\beta > 45°$) and $\cos\gamma > \frac{1}{\sqrt{3}}$

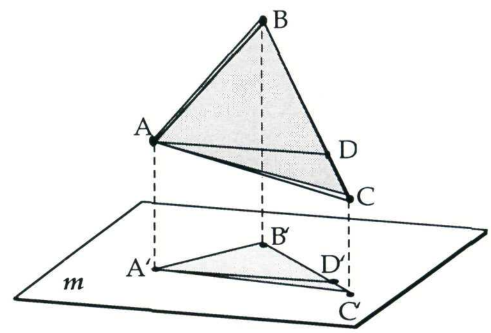
◊Problem 4 If the orthogonal projections of a segment onto the xy-, yz-, and xz-planes have the same length, do the orthogonal projections of the same segment onto the x-, y-, and z-axes have the same length ?

■ The Third Spatial Pythagorean Theorem (PT3) deals with the area, S, of a plane figure situated in space, and the areas, S_1, S_2, and S_3, of its projections onto three mutually perpendicular planes.

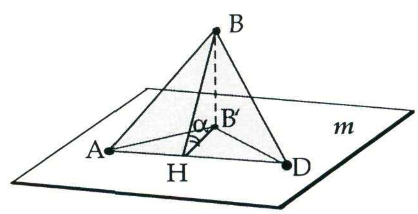
(i) The square of the area of an arbitrary bounded plane figure in space equals the sum of the squares of the areas of the projections of the figure onto three mutually perpendicular planes: $S^2 = S_1^{\,2} + S_2^{\,2} + S_3^{\,2}$.

If we consider the result for a triangle, then the theorem will extend to any plane figure since every plane polygon is a union of triangles. First, consider a little theorem (lemma) that we will need to prove PT3.

Lemma: Let ΔABC be located in space in a plane **p**. Let ΔA′B′C′ be the orthogonal projection of ΔABC onto another plane, *m*, where the angle between *p* and *m* is α. Then, Area(ΔA′B′C′) = Area(ΔABC)·cos α



It is enough to prove the statement only for a triangle with a side, AD, parallel to plane **m**. (If it does not have a side parallel to **m**, then the triangle could be split into two triangles, each of which has a side parallel to **m**. If the formula is proven for each triangular part, then it remains true for the whole triangle, too, since the sum of the areas of the projecting triangles times cos α equals the sum of the areas of the projected triangles.)



The proof of the particular case when a side of the projecting triangle,ABD, is parallel to the plane **m** is clear when we shift ΔABD down to meet the plane and ΔA′B′D′ along A′D′, so AD = A′D′. Also, height BH of ΔABD projects to height B′H of ΔA′B′D′. $\frac{B'H}{BH} = \cos\alpha$, in right triangle BB′H. We can divide the two area expressions:

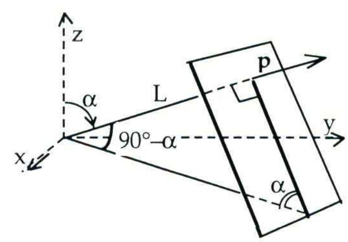
$$\text{Area}(\Delta ABD) = \frac{1}{2}AD\cdot BH \text{ and } \text{Area}(\Delta A'B'D') = \frac{1}{2}AD\cdot B'H.$$

$$\frac{\text{Area}(\Delta A'B'D')}{\text{Area}(\Delta ABD)} = \frac{B'H}{BH} = \cos\alpha, \text{ so that } \text{Area}(\Delta A'B'D') = \text{Area}(\Delta ABD)\cdot\cos\alpha$$

Lemma is proved! The lemma remains in force for any polygon which is the union of triangles.

[Now we can prove PT3. Use the coordinate planes, xy, yz and xz. Let the angles that plane p of the triangle makes with the coordinate planes be α, β, and γ. According to the lemma, the areas of the projections of the triangle onto the coordinate planes are: $S_1 = S \cdot \cos\alpha$, $S_2 = S \cdot \cos\beta$, $S_3 = S \cdot \cos\gamma$, where S is the area of the triangle . Let us notice that the line L perpendicular to the plane p makes the same angles, with the coordinate axes as the triangle does with the coordinate planes.

For example, the angle between plane p and the xy-plane is α, the line perpendicular to p is L, and the line perpendicular to the xy-plane is the z-axis; thus the angle between the line L and the z-axis is α, too. Likewise the angle between L and the x-, and y-axes are β and γ, respectively. Thus, by the Spacial Direction Cosines Corollary:



$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

thus,

$$S_1^2 + S_2^2 + S_3^2 = S^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

so that

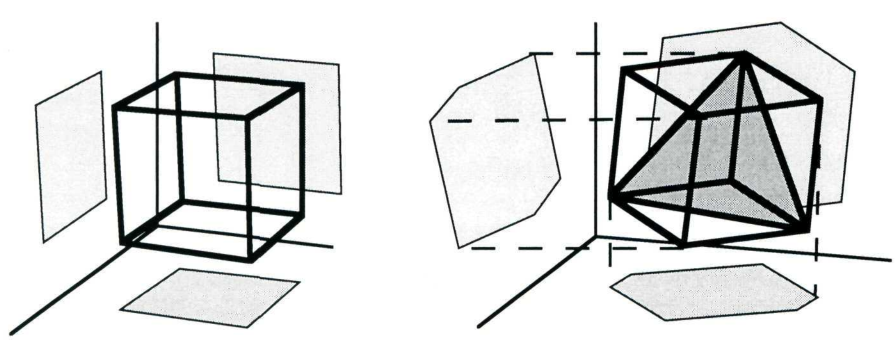
$$S^2 = S_1^2 + S_2^2 + S_3^2$$ and we are finished.]

◊Problem 5. (a) Consider a unit square situated in space. Let S_1, S_2, and S_3 be the projections onto the yx-, yz-, and xz-coordinate planes. How should the square be positioned to maximuze $S_1 + S_2 + S_3$? What is the maximum value?

(b) Solve the same problem for a regular hexagon and for a circle. Can you generalize your results to any plane figure?

(c) Project a unit cube onto the xy-plane. keep in mind that different faces will make different angles with the plane, and some faces will be behind others and not cast any shadow. What is the maximum possible area of the projection?

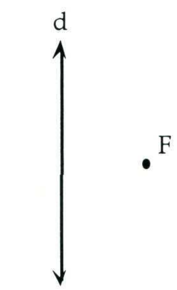
(d) If the projections of the cube onto all three coordinate planes have the same area, S, what possible values can S have?



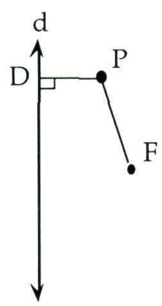
VARIABLE ECCENTRICITIES

by: Charles L. Hamberg and Susan K. Eddins
Illinois Mathematics and Science Academy

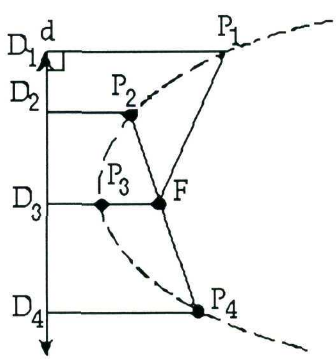
One approach used in defining and determining a conic begins with two simple geometric objects: a fixed line called a *directrix*, and a fixed point called a *focus*.



Suppose that we are given, a fixed line, d, and a fixed point, F. We can consider a conic to be the set (locus) of all points, P, on the plane where the ratio of its distance from the point F (PF) and its distance from the Line d (PD) is a constant.



That is, $\frac{FP}{PD} = e$ where e is a constant called the *eccentricity* of the conic.

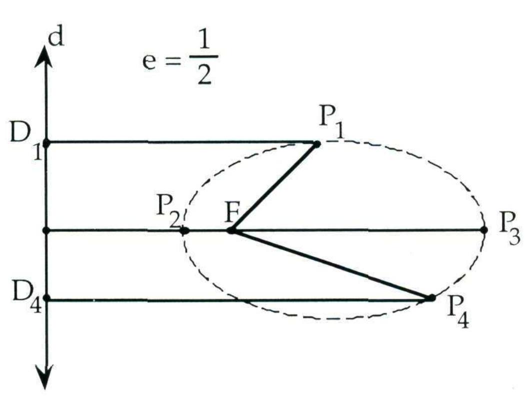


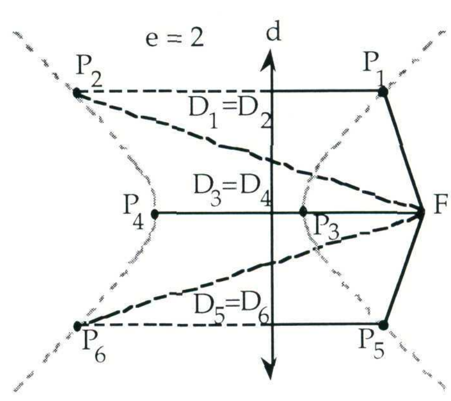
If we fix e to be equal to 1, we have a parabola.

Thus, if $\frac{FP_1}{P_1D_1} = \frac{FP_2}{P_2D_2} = \frac{FP_3}{P_3D_3} = \frac{FP_4}{P_4D_4} = 1$,

then P_1, P_2, P_3, and P_4 are points on a parabola.

We can change our conic by changing the fixed value of the eccentricity. If the value of the ratios is fixed as being less than 1 the points lie on an ellipse. If the value of e is fixed as being greater than 1 the points lie on a hyperbola.

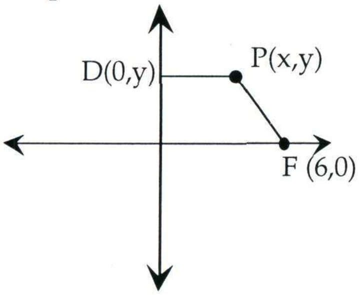




We note that in all previous examples the ratio **e** is fixed; a constant function. Suppose we were to let the value of **e** vary according to some equation; i.e. become a function of x. What types of curves might be produced? This question was first posed by a student, Mark Moody, at Adlai E. Stevenson High School, in the Fall of 1969. With the availability of the graphing calculator his question is more approachable today than it was then.

Let us consider some specific examples. In each Let $x = 0$ be a directrix and let $F(6,0)$ be the focus.

Example #1:



Suppose we locate all points $P(x,y)$ so that, $e(x) = x$ for all $x > 0$ that is $\frac{FP}{PD} = x$ for $x > 0$.

Thus, we have

$$\frac{FP}{PD} = \frac{\sqrt{(x-6)^2 + (y-0)^2}}{x} = x$$

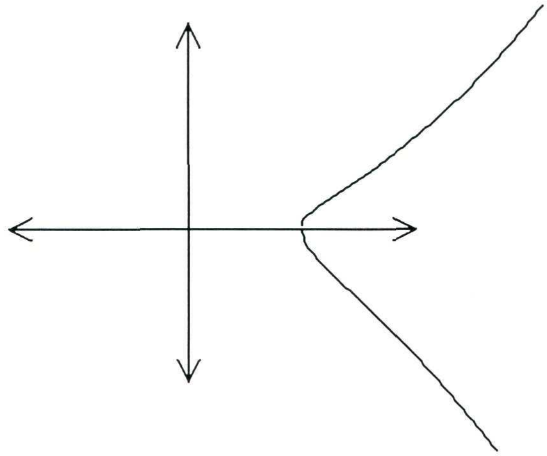
Squaring and rewriting gives us

$$y^2 = x^4 - (x-6)^2$$

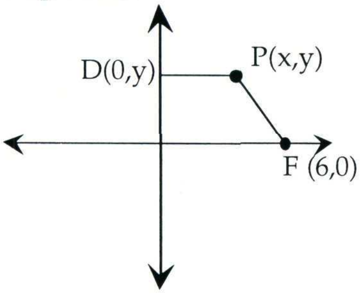
or

$$y = \pm\sqrt{x^4 - (x-6)^2}$$

1. A graphing calculator shows this graph of the relation for $x > 0$. For what domain is this relation defined?
2. What is the range?
3. What shape does the curve seem to approximate?
4. What would the graph be if the restriction $x > 0$ were removed?



Example #2:



Now consider $e(x) = \frac{FP}{PD} = x^2$ for $x > 0$.

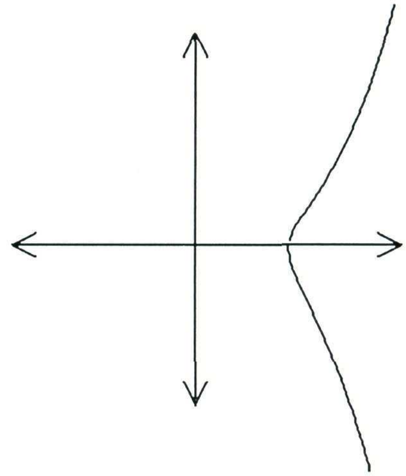
$$\frac{\sqrt{(x-6)^2 + (y-0)^2}}{x} = x^2$$

then, $y^2 = x^6 - (x-6)^2$

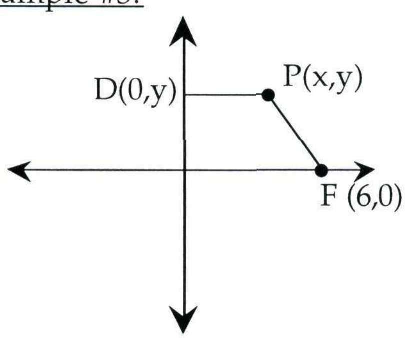
and

$$y = \pm\sqrt{x^6 - (x-6)^2}$$

1. Compare this graph and its domain with that found in Example #1.



Example #3:



Now look carefully at what happens when $e(x) = \dfrac{FP}{PD} = \dfrac{1}{x}$.

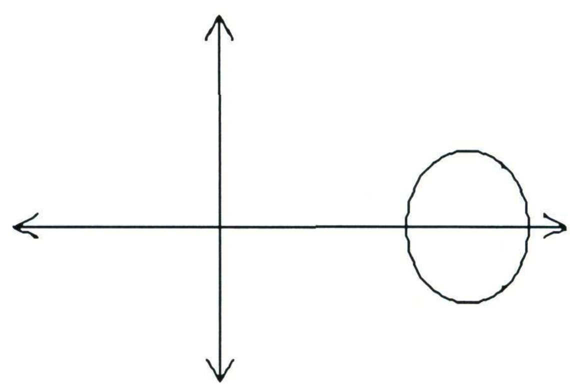
$$\frac{\sqrt{(x-6)^2 + (y-0)^2}}{x} = \frac{1}{x}$$

Squaring and simplifying we get

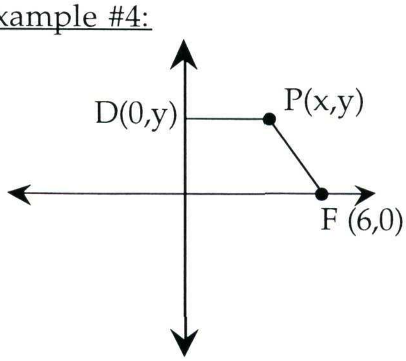
$$y^2 = 1 - (x-6)^2 \qquad \text{(a)}$$

$$y = \pm\sqrt{1-(x-6)^2} \qquad \text{(b)}$$

1. What is the domain for this relation when $x > 0$?
2. Consider equation (a). What relation does this equation describe?
3. Graph the relation with equation (b) on your graphing calculator using a window $\begin{array}{c} 3 \le x \le 9 \\ -2 \le y \le 2 \end{array}$



Example #4:



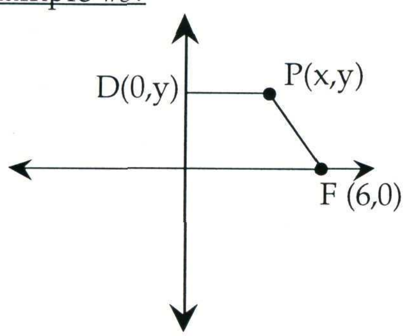
Repeat this process for $e(x) = \dfrac{FP}{PD} = \dfrac{1}{x^2}$.

See if you can "find" the graph!

1. Determine the domain for this relation.

Finally, consider a very different kind of eccentricity function.

Example #5:



What would the graph be if

$$e(x) = \frac{FP}{PD} = | \sin x | \text{ for } x > 0?$$

$$\frac{\sqrt{(x-6)^2 + (y-0)^2}}{x} = | \sin x |$$

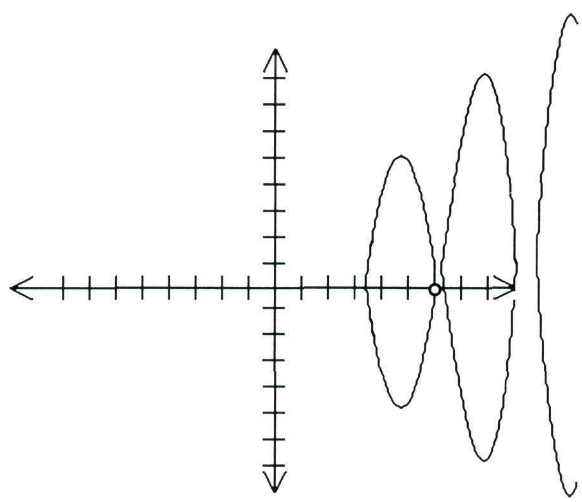
$$y^2 = x^2 \sin^2 x - (x-6)^2$$

or

$$y = \pm \sqrt{x^2 \sin^2 x - (x-6)^2}$$

Questions for investigation:

1. The graph of the relation for $x > 0$ is shown at the right. For what positive domain is the relation defined?
2. What is the range?
3. What shape, if any, does the curve seem to approximate?
4. Replace the ratio $| \sin x |$ by $\sin x$. How does that affect the graph?



If eccentricities are allowed to be functions of x then the resulting relations to be graphed require the use of technology. There is much here for students to explore and investigate.

The interested reader might try to explore what happens to the graph as x continues to increase. ✍

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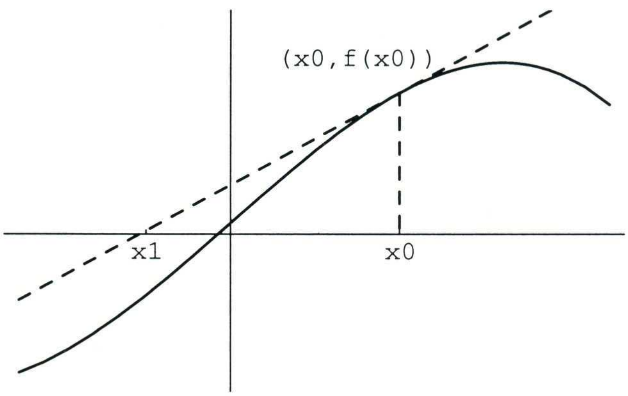
  E-mail: brenda@imsa.edu

IMSA

SEEING NEWTON'S METHOD

by: Ruth Dover
Illinois Mathematics and Science Academy

Newton's Method, or more appropriately the Newton-Raphson Method, is an algorithm for approximating the roots of functions. This follows a fairly simple procedure. The first step is shown in the diagram below.



tangent line:
$L(x) = f'(x_0)(x - x_0) + f(x_0)$

Given an equation $f(x) = 0$ to be solved, choose an approximation to the root to be found, and call it x_0. Write the equation of the tangent line to f at $x = x_0$. Find the x-intercept of the tangent line, and call this x_1.

Repeat the process by writing the equation of the tangent line to f at $x = x_1$. Label the new x-intercept x_2. Analytically, this can be given by the general formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Normally, this algorithm will generate a sequence of approximations $x_0, x_1, x_2, \ldots$, which become closer and closer to the desired root. This process may be continued as necessary to obtain the accuracy needed.

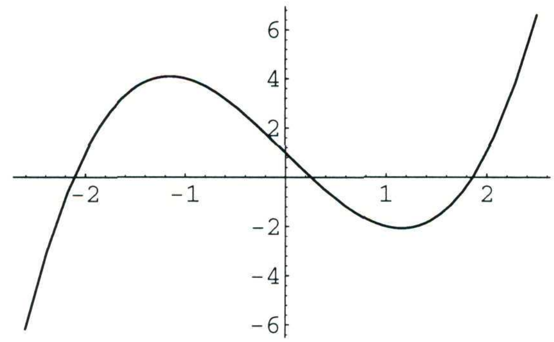
That's the basic approach. The sequence of approximations converges very quickly much of the time, so the method has been taught in calculus courses for years and years. However, its popularity was never very high without some sort of calculation hardware nearby. Even scientific calculators helped little, since the process was rather cumbersome with anything but the simplest of functions. Besides, the decimals often became more trouble than they were worth. Consequently, calculus students did a few problems dutifully while following the textbook, and then quickly forgot the topic. With the advent of graphing calculators, the root could be found for us, leaving Newton's Method again with little apparent relevance for many.

But wait! Watching the process turns out to be where much of the interesting mathematics comes into play.

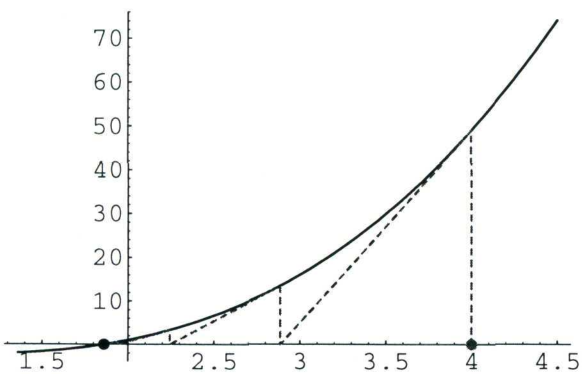
The software *Mathematica* uses Newton's Method for its "FindRoot" command. Here again, it assumes that your interest is in the result rather than the process. Another program, *The Joy of Mathematica*, used in conjuction with *Mathematica*, has a subroutine to show the process.

The graphics shown below were created using both pieces of software.

Let's begin with the equation $x^3 - 4x + 1 = 0$, with three real roots as shown.



To show the basic process, we will approximate the largest root. Here, as an initial (poor) guess, we choose $x_0 = 4$.

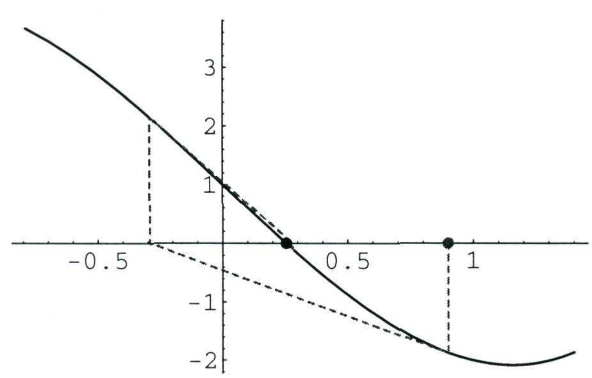


The large dot at x = 4 denotes the starting point. The dotted lines go up to the function, along the tangent line to the x-intercept, up to the function, etc. After three iterations, we can no longer see the path clearly since the iterations converged so quickly to the root, approximately 1.861. (Note that the vertical line is not the y-axis.)

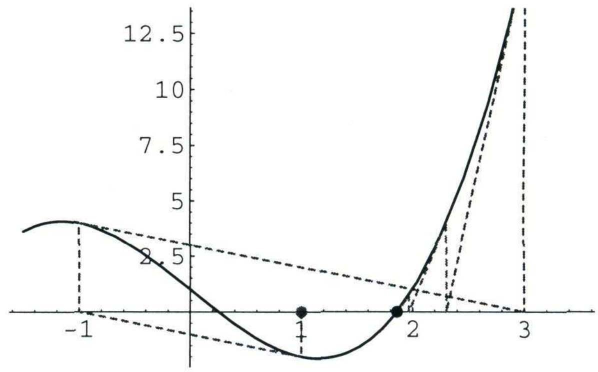
Beginning with any other x_0 which is greater than the largest root will produce a similar graph. A different number of iterations may be visible–either more or less–but the pattern of convergence will remain orderly.

By choosing other values for x_0 which are close to each of the other two roots, once can easily approximate these roots. But what happens when the values of x_0 are not as close to the roots? In other words, what happens if these values are not chosen so carefully?

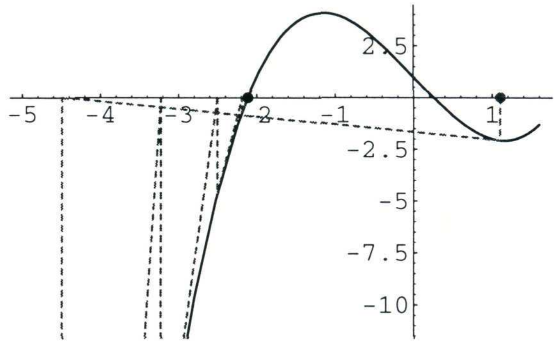
First, let $x_0 = 0.9$. The sequence converges to 0.254, the middle of the three roots.



Now, choose $x_0 = 1.0$. The tangent lines shoot first to the left and then back to the right, before settling down to an orderly pattern. Finally, the Method seems to converge to the largest root 1.861, just as in our original example with $x_0 = 4$.



Increase the initial value again by 1/10, so $x_0 = 1.1$. Parts of the dotted lines representing the tangent lines and vertical segments to the graph are not shown. Still, we can see that the first tangent line sends the next intercept far to the left. In the end, the smallest root is approximated as -2.115.



Small changes to the initial choice of x_0 led to the three different roots of the equation. The obvious question is to ask what happens between these values. What happens if $x_0 = 1.03$, for example? Or at 0.962? If you have access to these software packages, you are certainly encouraged to try these values. Both the issue of what root is finally approximated as well as the process or path required to reach the specific root raise interesting questions–and often unpredictable answers. Are there other sections of the real line which create similarly unpredictable results? Are there intervals where all initial inputs lead to a single root? Many such questions can be raised and lead to interesting investigations.

Though the visualization can add much to one's understanding, the sequence of values may easily be obtained with graphing calculators. To try this, enter the following short program: Newton.

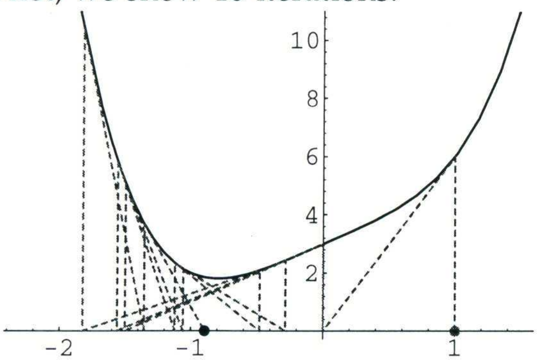
<u>TI-81</u>	<u>TI-82</u>	<u>TI-85</u>
: X - Y_1/NDeriv(Y_1,.001) → X	: X - Y_1/nDeriv(Y_1,X,X) → X	: x - y_1/nDeriv(y_1,x,x) → x
: Disp X	: Disp X	: Disp X

To use the program on the equation $f(x) = 0$, first enter the function f into Y_1. On the home screen, type in your choice for x_0 and press STO X and then Enter. Then run the program. Press Enter repeatedly to rerun the program and find subsequent iterations as desired. The first few outputs may vary quite erratically, but if the iteration is to converge, it will usually do so rather quickly. Watch the accuracy by paying attention to the stabilization of decimal places.

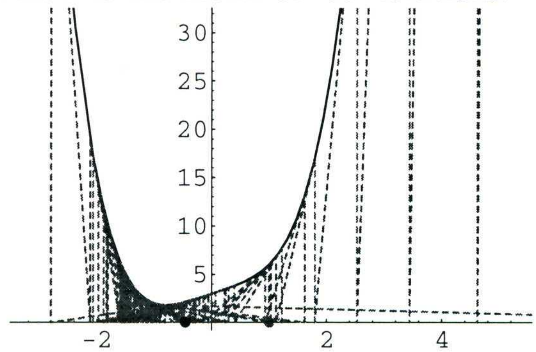
As another example, consider $x^4 + 2x + 3 = 0$. Graph this function to verify that it has no real roots. Clearly, we would hope that Newton's Method does not find roots which don't exist! Let $x_0 = 1$ on your graphing calculator to see what does happen.

The software shows us much here.

First, we show 10 iterations.



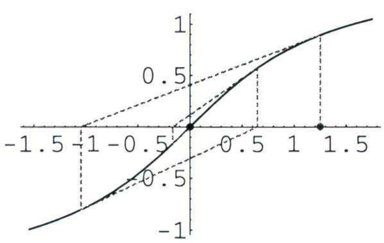
Below is the result of 80 iterations.

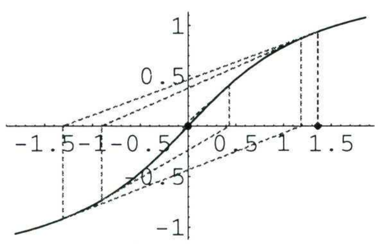


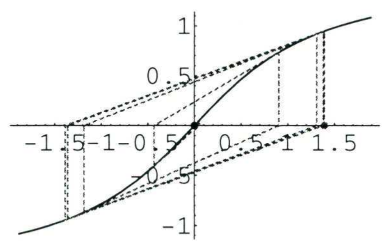
This is an example of chaotic behavior. Each output depends on the previous input, but there will be no convergence to any single value, and no discernible patterns.

For the third example, consider $f(x) = \text{Tan}^{-1}x$ with ten iterations.

Let $x_0 = 1.25$. Let $x_0 = 1.6$. Let $x_0 = 1.39$.







We seem to get a "parallelogram" forming, but the iteration eventually closes in on 0 as hoped. But try $x_0 = 1.392$. Or $x_0 = 1.396$. What happens?

Though these particular functions were chosen with certain basic characteristics, there is nothing special about the coefficients or the functions themselves. The reader is encouraged to create functions and to find initial inputs which generate interesting patters–or no patterns at all! ✍

The graphics depicted in this article were created using software published by the following:

- *Mathematica* , Wolfram Research, Inc., P.O. Box 6059, Champaign, IL 61826-6059, (217) 398-0700.

- *Joy of Mathematica* , Addison-Wesley Publishing Company, 1 Jacobs Way, Reading, MA 01867, (800) 552-2259.

GROUP TESTING: AN OPTIMIZATION PROBLEM FOR CALCULUS

by: Dan Teague
The North Carolina School of Science and Mathematics

Introduction

For the past several years, I have posed the following problem to my elementary calculus and mathematical modeling classes:

> Suppose that you have a large population (N) that you wish to test for a certain characteristic in their urine. Each test will be either positive or negative. Since the number of individuals to be tested is quite large, we can expect that the cost of testing will also be large. How can one reduce the number of test needed to screen everyone and thereby reduce the costs? If the urine could be pooled by putting G samples together and then testing the pooled sample, the number of tests required might be reduced. What is the relationship between the probability of an individual testing positive (p) and the group size (G) that minimizes the total number of tests requested?
>
> Use your solution to determine the number of tests required to find 100 individuals who will test positive in a population of 1,000,000.

The students spend approximately a week working in small groups on the problem. My role is to offer encouragement and suggestions as they develop their solutions. Over the year, the students have used many different approaches in solving this problem and developed several different models in the process. This article presents one of the more interesting solution paths. Although solutions need to be integral, it uses a continuous model and develops a theoretial optimum by assuming a simplifying, worst case situation; specifically, if a group tests positive, exactly one person in the group is positive.

The Full Lab Solution

The following are the variables that will be used in the solution:

N	Total number of persons
M	Maximum number of persons that can be tested at one time
P	Probability of person being positive
G_k	Size of kth group

The solution begins with the argument,

> The lab will likely have only a limited amount of equipment to use in the testing. Suppose the maximum number that can be tested at any one time is M. What would happen if, at each stage of the testing, we use all the available equipment, that is, test M groups each time.

Recall that we assume the worst case, that is, in any group that tests positive, exactly one person in the group is positive. This means that since we expect N•p individuals to test positive, we will have N•p groups testing positive as well.

If there are N persons to be tested, then the first round of tests would have a group size $G_1 = \frac{N}{M}$. Since we expect N•p individuals to test positive, under the worst case assumption, we would have N•p groups testing positive and therefore $N \bullet p \bullet G_1$ persons to be retested. The second testing would be in groups of size $G_2 = \frac{NpG_1}{M} = \frac{N^2 p}{M^2}$, resulting in $N \bullet p \bullet G_2$ persons needing retesting. Continuing, we find that the group size for the kth group is $G_k = \frac{N^k p^{k-1}}{M^k}$. We will have finished the testing when the group size is one, so the number of iterations of the procedure is determined by the value of k when $G_k = 1$. Solving $\frac{N^k p^{k-1}}{M} = 1$ for k, we find that

$$k = \frac{\ln(p)}{\ln\left(\frac{Np}{M}\right)}.$$

In this scenario, we have k iterations of M tests, so the total number of tests is given by $T(M) = M \cdot k = M\left[\frac{\ln(p)}{\ln\left(\frac{Np}{M}\right)}\right]$. We have assumed that we should use all of the testing facilities available, but this assumption is contradicted by the model. The function T(M) defined above has a minimum value, indicating an optimal value for M. What value of M minimizes the total number of tests? To minimize T, we differentiate with respect to M to find

$$\frac{dT}{dM} = \frac{\ln\left(\frac{Np}{M}\right)\ln(p) - M\ln(p)\left(\frac{M}{Np}\right)\left(-\frac{Np}{M^2}\right)}{\ln^2\left(\frac{Np}{M}\right)}.$$

The derivative is defined for all M>0, and setting the derivative equal to zero and simplifying yields the equation

$$\ln\left(\frac{Np}{M}\right)+1=0$$

Solving for M we find that

$$M=Npe.$$

If possible, to minimize the number of tests, perform M=Npe tests on each iteration with a group size on the kth iteration of

$$G_k=\frac{N^k p^{k-1}}{M^k}=\frac{N^k p^{k-1}}{(Npe)^k}=\frac{1}{pe^k}.$$

The total number of tests required using this procedure is given by

$$T=(Npe)\left(\frac{\ln(p)}{\ln\left(\frac{Np}{Npe}\right)}\right)=(Npe)(-\ln(p))\text{ total tests.}$$

In our problem, with 1,000,000 people and 100 testing positive, we would need only $(Npe)(-\ln(p)=(1{,}000{,}000)(.0001)(e)(-\ln(.0001))\approx 2{,}500$ tests!

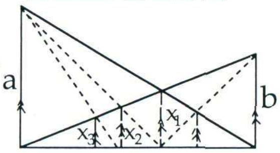
Students are always surprised to see e show up in the solution. Of course, while this theoretical result is pleasing, it may not be realizable, for $G_k=\frac{1}{pe^k}$ may be too many specimen to handle in a single group.

The History of "Noah Sheets"

The following four pages include a collection of interesting and useful mathematical formulas and relationships that were originally gathered together by IMSA alumnus, Noah Rosenberg, when he was a student at the Academy. He was an enthusiaistic participant in math competitions and complied them for use by our math team. His hand written notes have been edited and enhanced by Mr. George Milauskas (IMSA mathematics faculty). The resulting materials are affectionately known as the "Noah Sheets". Our hopes are that you will find some worthwhile ideas for use in math classes, mathlete training and problem solving.

Triangles

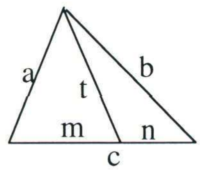
Two Pole Problem



$\frac{1}{x_1} = \frac{1}{a} + \frac{1}{b}$ so

$x_1 = \frac{a \cdot b}{a + b}$ $\quad x_k = \frac{a \cdot b}{a + kb}$

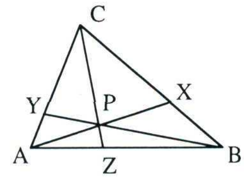
Stewart's Theorem



$a^2n + b^2m = t^2c + m \cdot n \cdot c$

(Proven by using Law of Cosines twice)

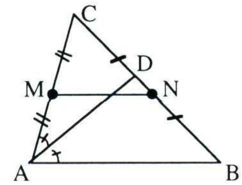
Ceva's Theorem



$\frac{AZ}{ZB} \bullet \frac{BX}{XC} \bullet \frac{CY}{YA} = 1, \quad \frac{PX}{AX} + \frac{PY}{BY} + \frac{PZ}{CZ} = 1$

$\Leftrightarrow$ AX, BY, & CZ are concurrent

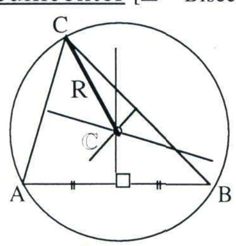
Basic Theorems



AC:AB = CD:DB ($\angle$bis thm)

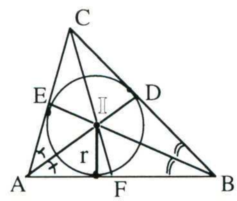
MN $\parallel$ & $\frac{1}{2}$ AB [Midline Thm]

Circumcenter [$\perp$ – Bisectors]



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

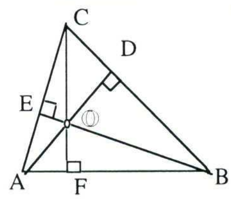
Incenter [$\angle$ - Bisectors]



Area (ΔABC) $= \frac{1}{2}$ r P

$=$ r$\cdot$s (s=semi-perimeter)

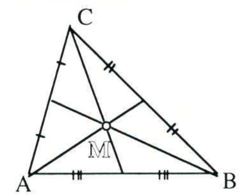
Orthocenter [Altitudes]



Watch for similar triangles.

eg: ΔADB ~ ΔCFB

Centroid [Medians]

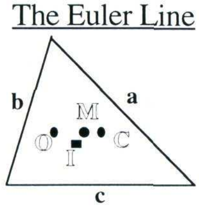


All six areas are equal.

M splits each median in ratio 2:1.

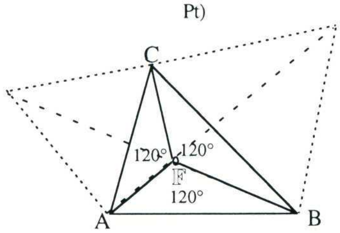
Coordinates of M = avg of vertices.

The Euler Line



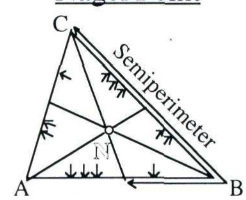
O, M, & C are collinear, such that OM : MC = 2:1 and $9 \cdot (OC)^2 = a^2 + b^2 + c^2$

Fermat Point (Equiangular Pt)



The sum AF + BF + CF is a minimum. (Found by, putting equilatersl Δ's on sides)

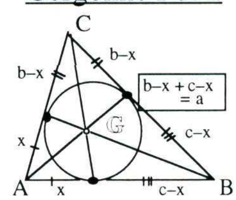
Nagel Point



[Joins Semi-Perimeter Pts to Vertices]

Notice resulting $\cong$ segments

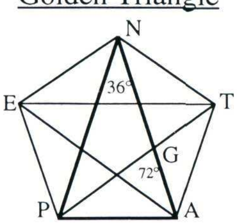
Gergonne Point



[Tangency Pts to Vertices]

Notice segments & "walkaround" labelling.

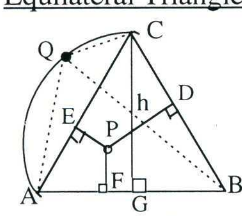
Golden Triangle



In a regular pentagon,

PN:PA = PG:GA $= \frac{1+\sqrt{5}}{2}$

Equilateral Triangles

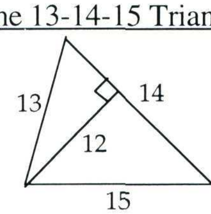


Sum of dist from any **P** to sides = **h**.

Any **Q** on Circum-$\odot$: QB = QC + QA

ΔCGB, has sides in a ratio, $1 : \sqrt{3} : 2$

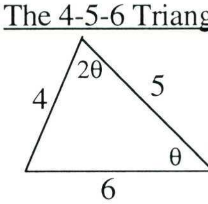
The 13-14-15 Triangle



[An altitude and three sides are consecutive integers.]

Area = 84, r = 4, R $= \frac{65}{8}$

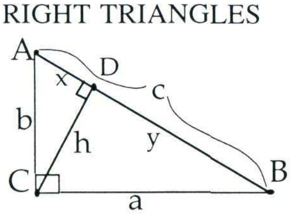
The 4-5-6 Triangle



One angle is twice the other.

Area $= 6\sqrt{6}$

RIGHT TRIANGLES



$(a + b)^2 = c^2 + 4(\text{Area})$

$\frac{y}{a} = \frac{a}{c} \Rightarrow a^2 = y \cdot c$

$\frac{x}{b} = \frac{b}{c} \Rightarrow b^2 = x \cdot c$

$\frac{x}{h} = \frac{h}{y} \Rightarrow h^2 = x \cdot y$

$a \cdot b = c \cdot h$

$a^2 + b^2 = c^2$ Pythagorean Thm

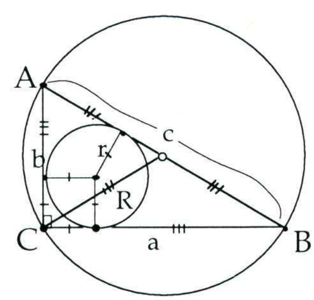
$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2}$

$2 \cdot r = a + b - c$

$2 \cdot R = c \quad \frac{a+b}{2} = R + r$

median $= \frac{1}{2}$ hypotenuse

$(m^2 - n^2,\ 2mn,\ m^2 + n^2)$ is Pythagorean triple for m,n positive integers



A Triangle And Its Circles: PROPERTIES

Δ ABC has sides: **c**, **b**, and **a**, and angles A, B, and C.

The radii of the:

Inscribed circle, **r**.

The three escribed circles: $\mathbf{r_a}$, $\mathbf{r_b}$, & $\mathbf{r_c}$

and the circumscribed circle, **R**.

The area of the triangle is **K**.

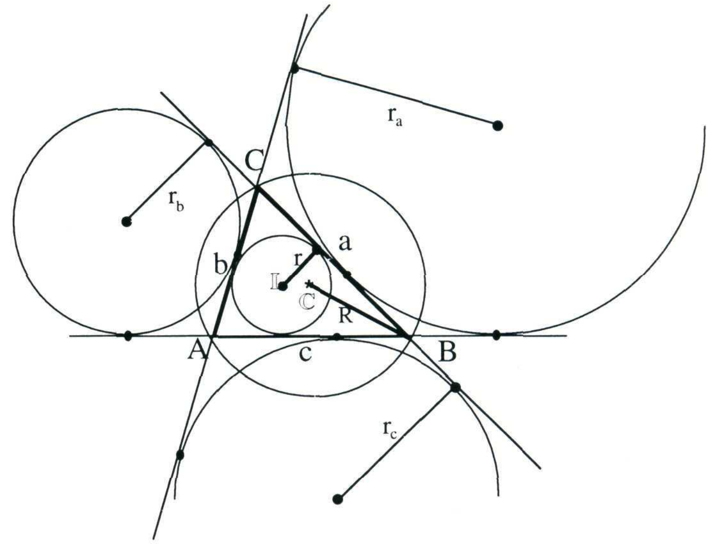
The semiperimeter is **S**.

Drawn to sides **a**, **b**, and **c**, respectively:

Let $\mathbf{m_a}$, $\mathbf{m_b}$, & $\mathbf{m_c}$ be the medians.

Let $\mathbf{t_a}$, $\mathbf{t_b}$, & $\mathbf{t_c}$ be the angle bisectors.

Let $\mathbf{h_a}$, $\mathbf{h_b}$, & $\mathbf{h_c}$ be the altitudes.



The following relationships are true for triangles as labeled above:

$K = \sqrt{S(S-a)(S-b)(S-c)}$ Heron's Formula

$K = \frac{1}{2}\cdot a\cdot b\cdot \sin C = \frac{a^2 \sin B\cdot \sin C}{2\sin A} = r\cdot S = \frac{a\cdot b\cdot c}{4\cdot R}$

$R = \frac{a\cdot b\cdot c}{4\cdot K}$ $\quad 2\cdot R\cdot r = \frac{a\cdot b\cdot c}{a+b+c}$ $\quad 2\cdot r \le R$ in all Δ's

$t_c = \frac{2\cdot a\cdot b\cdot \cos\frac{C}{2}}{a+b} = \frac{2\sqrt{a\cdot b\cdot S\cdot(S-c)}}{a+b}$

$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$ $\quad r^2 = \frac{(s-a)(s-b)(s-c)}{s}$

$c^2 = 2a^2 + 2b^2 - 4m_c^2$ and its permutations

Law of Cosines:
$c^2 = a^2 + b^2 - 2\cdot a\cdot b\cos C$ (& permutations)

Law of Sines:
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Law of Tangents: $\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$

Other

Point-Line(plane) Distance

between (x_0, y_0) and line $ax + by + c = 0$:

dist is: $\frac{|a\cdot x_0 + b\cdot y_0 + c|}{\sqrt{a^2+b^2}}$

between (x_0, y_0, z_0) and line $ax + by + c\cdot z + d = 0$:

is: $\frac{|a\cdot x_0 + b\cdot y_0 + c\cdot z_0 + d|}{\sqrt{a^2+b^2+c^2}}$

also dist $= \frac{a\cdot b\cdot c}{a\cdot b + b\cdot c + a\cdot c}$

Logarithms

$\log_b N = p \Leftrightarrow b^p = N$

$\log_b N = \frac{\log_a N}{\log_a b}$ "change base"

$\log\frac{m\cdot n}{q} = \log m + \log n - \log q$

$\log N^p = p \log N$

$\log_b a = \frac{1}{\log_a b}$

Series: $S = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

•Arithmetic: Constant Difference $d = a_{n+1} - a_n$

$a_n = a_1 + (n-1)\cdot d$ & $S_n = \frac{n}{2}(a_1 + a_n)$

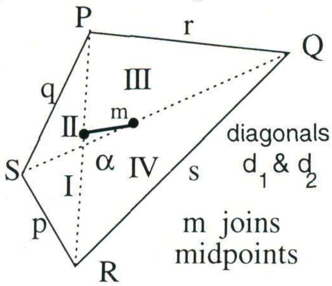
•Geometric: Constant ratio $r = \frac{a_{n+1}}{a_n}$

$a_n = a_1\cdot r^{n-1}$, $S_n = \frac{a - a\,r^{n-1}}{1-r}$ & $S_\infty = \frac{a_1}{1-r}$

$(-1 < r < 1)$

If you find any errors, or have any worthy additions to "The Noah Sheets" please contact George Milauskas, Mathematics Coordinator at the Illinois Mathematics and Science Academy, 1500 Sullivan Rd, Aurora, Illinois, 60506 (708)907-5965: E-mail: geom@imsa.edu

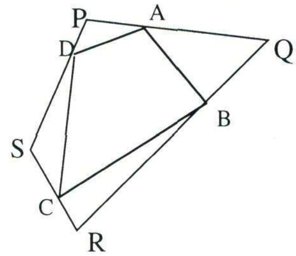
Quadrilateral Properties: **K** = Area, **r** = inradius, **R** = circumradius, **P** = perimeter, **S** = semiperimeter



Areas: $A_I \cdot A_{III} = A_{II} \cdot A_{IV}$

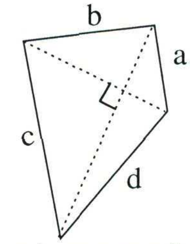
$K_{PQRS} = \frac{1}{2} d_1 d_2 \sin\alpha$

$p^2 + q^2 + r^2 + s^2 = d_1^2 + d_2^2 + (2m)^2$



IF A,B,C,D are midpoints, ABCD is a parallelogram.

If $\frac{PA}{AQ} = \frac{QB}{BR} = \frac{RC}{CS} = \frac{SD}{DP} =$ n then the ratio of areas, **K**(ABCD) : **K**(PQRS) = $\frac{n^2+1}{(n+1)^2}$



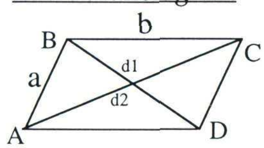
If Diagonals are perpendicular,

K= $\frac{1}{2}$(diag 1)(diag 2)

$a^2 + c^2 = b^2 + d^2$

⇔ (if and only if) the diagonals are ⊥..

In a Parallelogram



$2(a^2 + b^2) = d_1^2 + d_2^2$

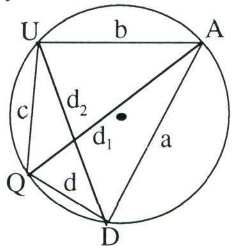
$BD = a^2 + b^2 - 2ab\cos A$

$K = a \cdot b \cdot \sin\angle A$.

$\frac{d_1}{d_2} = \frac{ab + cd}{ad + bc}$

(2nd Ptolemy's Theorem)

Cyclic Quadrilaterals

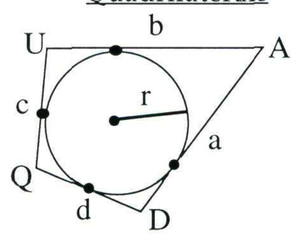


$\angle A + \angle Q = \angle U + \angle D = 180°$

$d_1 \cdot d_2 = a \cdot c + b \cdot d$ (Ptolemy)

$K = \sqrt{(S-a)(S-b)(S-c)(S-d)}$

Circumscribed Quadrilaterals

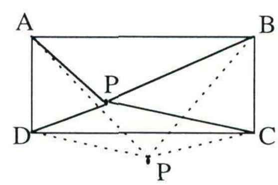


a+c = b+d ⇔ QUAD has in⊙

$r = \frac{K}{a+c}$ and $K = \frac{1}{2}\, r \cdot P$

If QUAD is both inscribed. and circumscr., then $K = \sqrt{a \cdot b \cdot c \cdot d}$

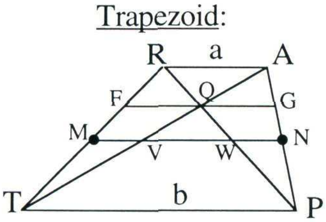
The Rectangle



For any point, P, & rectangle ABCD

$(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$

Trapezoid:



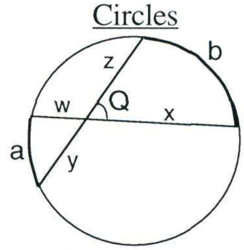
If MN = median, MN= $\frac{a+b}{2}$

$VW = \frac{b-a}{2}$ $\quad FG = \frac{2 \cdot a \cdot b}{a+b}$

If $\overline{MN}$ is ***any*** parallel,

$MN = \frac{RM \cdot b + MT \cdot a}{RT}$

Circles

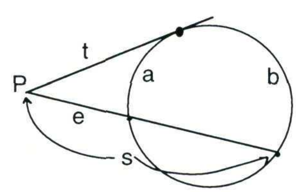


Angle-Arc Property

$Q = \frac{b + a}{2}$

Power Theorem

$w \cdot x = y \cdot z$

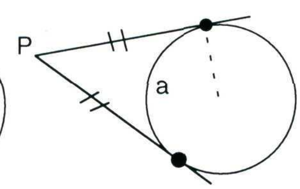


Angle-Arc Property

$P = \frac{b - a}{2}$

Power Theorem

$t^2 = s \cdot e$



A tangent to a circle is perpendicular to a radius.

Two tangents to a circle from an outside point are equal.

tan-tan angle = supp of inner arc. P + a = 180°

Volume and Surface Areas of Solids:

Prismatic solids: (prism, box, cylinder)
[Congruent cross sections]

Lateral Area = (base perimeter)(height)

Total Area = lateral area + 2 bases

Volume = (Area of Base)(height)

Pointed Solids: (pyramid, cone)
[Linearly related cross sections]

Lateral Area (add lat faces), $LA_{(cone)} = \pi \cdot r \cdot l$
l = lateral edge (slant height)

Total Area = lateral area + one base

Volume = $\frac{1}{3}$(Area of Base)(height)

Spheres:

Total Surface Area = $4 \cdot \pi \cdot r^2$

Volume = $\frac{4}{3}\pi r^3$

Ellipsoid: Volume = $\frac{4}{3} a \cdot b \cdot c \cdot \pi$

Prismoidal Volume Formula: $V = \frac{h}{6}(B_1 + 4M + B_2)$

[For solids with quadratically related cross sections, height h, upper bases B_1, B_2 and mid section M]

Trigonometry:

$$\sin A = \frac{\text{opp leg}}{\text{hypotenuse}}$$

$$\cos A = \frac{\text{adj leg}}{\text{hypotenuse}}$$

$$\tan A = \frac{\text{opp leg}}{\text{adj leg}} = \frac{\sin A}{\cos A}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opp leg}} = \frac{1}{\sin A}$$

$$\sec A = \frac{\text{hypotenuse}}{\text{adj leg}} = \frac{1}{\cos A}$$

$$\cot A = \frac{\text{adj leg}}{\text{opp leg}} = \frac{1}{\tan A}$$

Values to Memorize:

$$\sin 30° = \frac{1}{2} = \cos 60°$$

$$\cos 30° = \frac{\sqrt{3}}{2} = \sin 60°$$

$$\tan 30° = \frac{\sqrt{3}}{3} = \cot 60°$$

$$\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$$

$$\tan 45° = 1$$

$$\sin 15° = \frac{\sqrt{6} - \sqrt{2}}{4} = \cos 75°$$

$$\cos 15° = \frac{\sqrt{6} + \sqrt{2}}{4} = \sin 75°$$

$\tan 15° = 2 - \sqrt{3}$, $\tan 75° = 2 + \sqrt{3}$

Golden rectangle & regular pentagon.

$$\sin 18° = \cos 72° = \frac{\sqrt{5} - 1}{4}$$

$$\cos 36° = \sin 54° = \frac{\sqrt{5} + 1}{4}$$

Pythagorean Identities

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$1 + \cot^2 A = \csc^2 A$$

Odd-Even Functions:

$\sin(-A) = -\sin(A)$

$\cos(-A) = \cos(A)$

$\tan(-A) = -\tan(A)$

Complements A & B

$\sin^2 A + \sin^2 B = 1$

$\sin A = \cos B$, etc

Sum & Difference Identities

$$\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B$$

$$\cos(A \pm B) = \cos A \cdot \cos B \mp \sin A \cdot \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$$

Double Angle Identities

$$\sin 2A = 2 \sin A \cdot \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

or $= 1 - 2\sin^2 A = 2\cos^2 A - 1$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Sum to Product

$$\sin A + \sin B = 2 \sin\frac{A+B}{2} \cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2 \sin\frac{A-B}{2} \cos\frac{A+B}{2}$$

$$\cos A + \cos B = 2 \cos\frac{A+B}{2} \cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin\frac{A+B}{2} \sin\frac{A-B}{2}$$

$$\tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cdot \cos B}$$

Triple Angle Identities

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{\tan A \cdot (\tan^2 A - 3)}{3 \tan^2 A - 1}$$

Half Angle Formulas

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$$

$$\tan\frac{A}{2} = \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A}$$

Product to Sum

$$\sin A \cdot \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

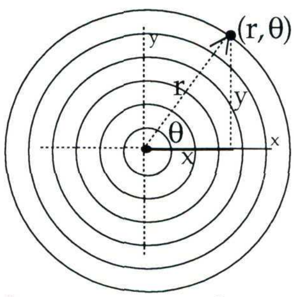
$$\cos A \cdot \cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sin A \cdot \cos B = \tfrac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\tan A \cdot \tan B = \frac{\cos(A-B) - \cos(A+B)}{\cos(A-B) + \cos(A+B)}$$

Polar Coordinates

Points are represented interms of (r,θ) rather than (x,y)



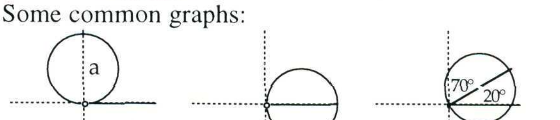
$$x^2 + y^2 = r^2$$

$$\tan\theta = \frac{y}{x}$$

$$x = r\cos\theta$$

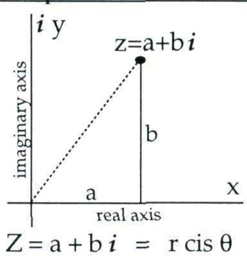
$$y = r\sin\theta$$

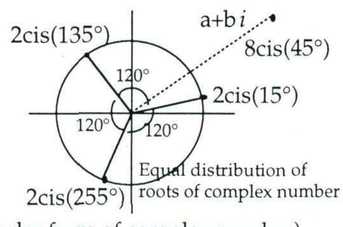
Some common graphs:



$r = a \sin\theta$ $r = a\cos\theta$ $r = a\sin(\theta + 70°)$ $r = a\cos(\theta - 20°)$

Complex Numbers, DeMoivre's Thm, Euler's Thm & CIS





$Z = a + bi = r \text{ cis } \theta$ (polar form of complex number)

The magnitude, $r = |a + bi| = \sqrt{a^2 + b^2}$

$e^{i\theta} = \cos\theta + i\sin\theta = \text{cis }\theta$ (Euler)

$\text{cis}(A + B) = \text{cis } A \cdot \text{cis } B$ $\quad$ $\text{cis}(A - B) = \dfrac{\text{cis } A}{\text{cis } B}$

DeMoivre's Theorems: (see illustration above)

$(a+bi)^n = (r \text{ cis } \theta)^n = r^n \text{ cis}(n \cdot \theta)$ for n = pos int

$\sqrt[n]{r \cdot \text{cis } \theta} = \sqrt[n]{r} \cdot \text{cis}\left(\frac{2\pi k + \theta}{n}\right)$ for $k = 0,1,2,3\ldots,n-1$

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Illinois Mathematics and Science Academy
1500 West Sullivan Road
Aurora, Illinois 60506-1000
(708) 907-5000

Printing made possible entirely by AT&T through its People First Gold Star Program.